16.6 Parametric Surfaces and their Areas

We can describe surfaces similarly to spacecurves: use two parameters u, v instead of just t.

Definition. Let x, y, z be functions of two variables: (u, v) describes a point in the domain $D \subseteq \mathbb{R}^2$. The *parametric surface* with *co-ordinate functions* x, y, z is the collection S of points with position vectors

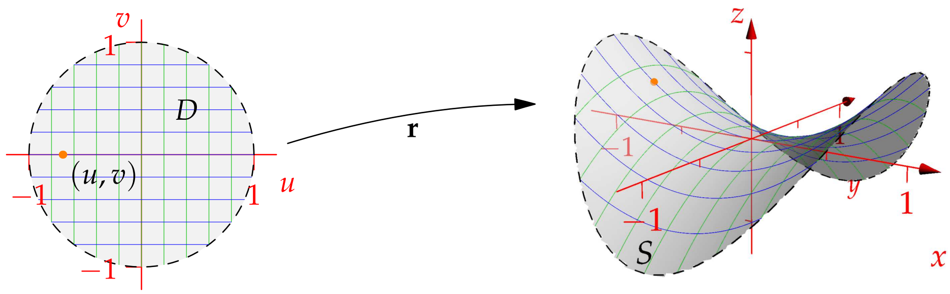
$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

We often speak (and think) about a parametrized surface as the function $\mathbf{r} : D \to \mathbb{R}^3$.
The *grid lines* are the two families of curves (on S) for which either u or v is constant.

Example 1 Suppose $D = \{(u, v) : u^2 + v^2 \leq 1\}$ is the unit disk in (u, v)-land and define $\mathbf{r} : D \to \mathbb{R}^3$ by

$$\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$$

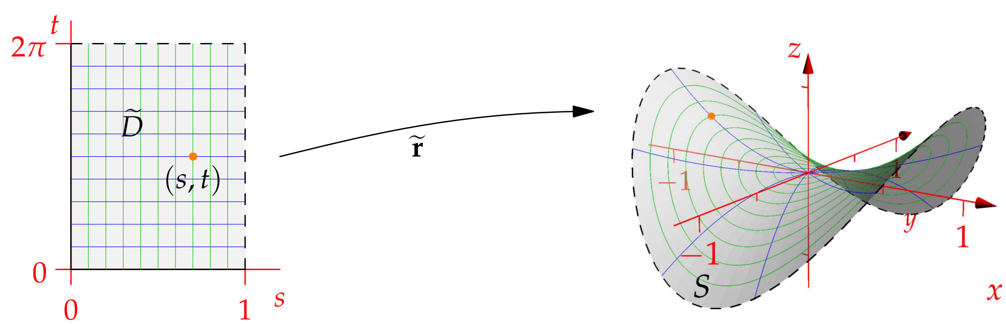
The surface S parametrized by $\mathbf{r}$ is drawn below along with the grid lines of the parametrization.[1] We recognize it as part of the saddle surface $z = x^2 - y^2$.



In fact the graph of any function $z = f(x, y)$ can be parametrized in this fashion. Geometrically, D is the result of projecting the surface onto the (x, y)-plane. One may similarly parametrize functions of other variables, e.g. $y = g(x, z)$ by projecting onto the (x, z)-plane.
A *surface* (as a subset of $\mathbb{R}^3$) may be parametrized in infinitely many ways! For instance, we could also parametrize the saddle-surface using what are essentially polar co-ordinates:

$$\tilde{\mathbf{r}}(s, t) = s\cos t\,\mathbf{i} + s\sin t\,\mathbf{j} + s^2\cos 2t\,\mathbf{k}, \quad \text{with domain} \quad \widetilde{D} = \{(s, t) : 0 \leq s < 1,\ 0 \leq t < 2\pi\}$$

The new parametrization $\tilde{\mathbf{r}} : \widetilde{D} \to \mathbb{R}^3$ has a different formula and domain, but the same range S.



[1] The co-ordinates (u, v) are essentially just (x, y), so we might write $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$ without introducing u, v at all. This approach is often preferable when using standard co-ordinates as it helps visualize things. The danger is that the *parameters* u, v have a different domain D from the *co-ordinate functions* x, y, z ($\mathbb{R}^3$). Be clear if x, y, z (or polar co-ordiantes r, θ, etc.) are might mean multiple things.

Regular Surfaces and Tangent Planes

To do calculus, we want to compute tangent planes. This places a restriction on our parametrizations. Suppose a surface S is parametrized by a differentiable function $\mathbf{r}(u, v)$. Since differentiating with respect to u means holding v constant, it follows that $\mathbf{r}_u(u_0, v_0)$ is tangent to the grid line $v = v_0$ on the surface. Indeed:

The partial derivatives $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$ are tangent to the surface

Definition. A point with position vector $\mathbf{r}(u_0, v_0)$ is *regular* if the tangent vectors arising from the parametrization are *non-parallel*: that is, $(\mathbf{r}_u \times \mathbf{r}_v)(u_0, v_0) \neq \mathbf{0}$. At a regular point:

- The *tangent plane* is spanned by the *tangent vectors* $\mathbf{r}_u$ and $\mathbf{r}_v$.
- The *unit normal vector* is $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$.

Example 1 (cont.) The parametrization $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$ is regular at all points:

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix} \implies \mathbf{n} = \frac{1}{\sqrt{1 + 4u^2 + 4v^2}} \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}$$

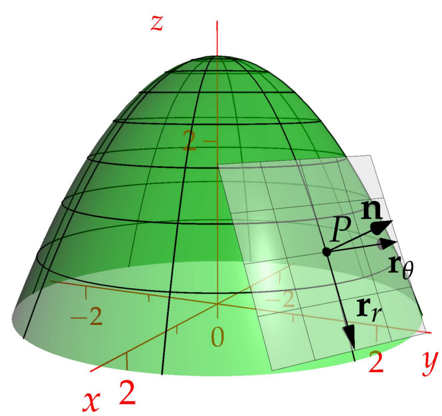
The polar parametrization $\tilde{\mathbf{r}}(s, t) = s \cos t\,\mathbf{i} + s \sin t\,\mathbf{j} + s^2 \cos 2t\,\mathbf{k}$ is regular except at the origin:

$$\tilde{\mathbf{r}}_s = \begin{pmatrix} \cos t \\ \sin t \\ 2s \cos 2t \end{pmatrix}, \quad \tilde{\mathbf{r}}_t = \begin{pmatrix} -s \sin t \\ s \cos t \\ -2s^2 \sin 2t \end{pmatrix} \implies \tilde{\mathbf{n}} = \frac{1}{\sqrt{1 + 4s^2}} \begin{pmatrix} -2s \cos t \\ 2s \sin t \\ 1 \end{pmatrix}, \ s \neq 0$$

From now on, whenever we parametrize using standard rectangular or polar co-ordinates, we'll use the relevant symbols rather than u, v.

Example 2 The paraboloid $z = 3 - \frac{1}{2}(x^2 + y^2)$ may be parametrized in polar co-ordinates: $x = r \cos \theta$, $y = r \sin \theta$.

$$\mathbf{r}(r, \theta) = r \cos \theta\,\mathbf{i} + r \sin \theta\,\mathbf{j} + (3 - \tfrac{1}{2} r^2)\,\mathbf{k}$$



This is regular except when $r = 0$:

$$\mathbf{r}_r = \begin{pmatrix} \cos \theta \\ \sin \theta \\ -r \end{pmatrix}, \quad \mathbf{r}_\theta = r \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}$$

As pictured, at $P = (2, \sqrt{3}, 1)$ with position vector $\mathbf{r}(2, \frac{\pi}{3})$,

$$\mathbf{r}_r \times \mathbf{r}_\theta = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{3} \\ -4 \end{pmatrix} \times \begin{pmatrix} -\sqrt{3} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2\sqrt{3} \\ 2 \end{pmatrix} \implies \mathbf{n} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ \sqrt{3} \\ 1 \end{pmatrix}$$

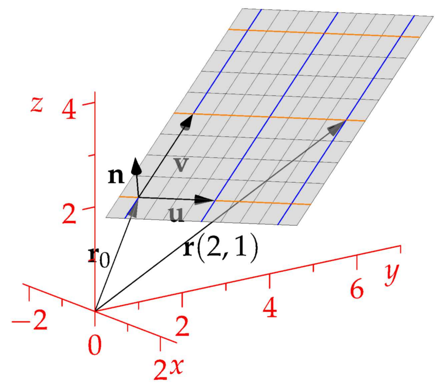
The tangent plane therefore has equation

$$\begin{aligned}(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0 &\implies 2(x - 2) + 2\sqrt{3}(y - \sqrt{3}) + (z - 1) = 0 \\ &\implies 2x + 2\sqrt{3}y + z = 11\end{aligned}$$

Standard Examples of Parametrized Surfaces

Planes Let $\mathbf{r}_0, \mathbf{u}, \mathbf{v}$ be constant vectors, where $\mathbf{u}, \mathbf{v}$ are non-parallel. The plane through a point with position vector $\mathbf{r}_0$, parallel to $\mathbf{u}$ and $\mathbf{v}$ has parametrization

$$\mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{u} + v\mathbf{v}, \quad u, v \in \mathbb{R}$$



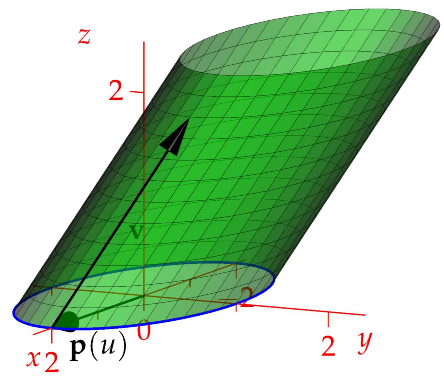
This is regular everywhere: indeed the plane is its own tangent plane at all points! Equivalently, $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ where $\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$ is the unit normal vector. The picture shows

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{n} = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

The grid lines are parallel to $\mathbf{u}, \mathbf{v}$: those in the picture correspond to $u = 0, 1, 2$ and $v = 0, 1, 2$.

Cylinders A *cylinder* is a parametric surface

$$\mathbf{r}(u,v) = \mathbf{p}(u) + v\,\mathbf{v}, \quad u_0 < u < u_1, \quad v \in \mathbb{R},$$



where $\mathbf{p}(u)$ describes a spacecurve and $\mathbf{v}$ is a constant vector: the parametrization is regular if $\mathbf{v}$ is never parallel to $\mathbf{p}'(u)$.

The grid lines come in two families: copies of the spacecurve translated in the $\mathbf{v}$-direction and lines parallel to $\mathbf{v}$.

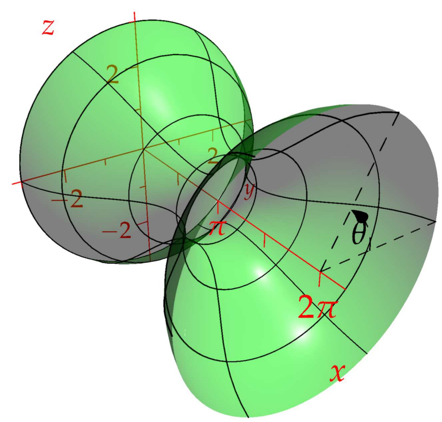
The picture shows cylinder with elliptic cross-sections:

$$\mathbf{r}(u,v) = \begin{pmatrix} 2\cos u \\ \sin u \\ 0 \end{pmatrix} + v \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

A *right-circular cylinder* is when $\mathbf{p}(u)$ describes a circle and $\mathbf{v}$ is perpendicular to the plane of the circle.

Surfaces of Revolution Rotate a curve $y = f(x)$ for $a \le x \le b$ around the x-axis. If x measures along the x-axis and θ is the polar angle in the yz-plane, we obtain

$$\mathbf{r}(x,\theta) = x\,\mathbf{i} + f(x)(\cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k})$$



This is regular everywhere if $f(x) > 0$: as parametrized, the unit normal $\mathbf{n}$ points *inwards* (think about it...).

The plot is the result when $f(x) = 2 + \cos x$, thus

$$\mathbf{r}(x,\theta) = \begin{pmatrix} x \\ (2+\cos x)\cos\theta \\ (2+\cos x)\sin\theta \end{pmatrix}, \quad 0 \le x, \theta \le 2\pi$$

The grid lines come in two families: copies of the original curve rotated around the x-axis, and circles centered on the x-axis. We could also rotate around any other axis.

Spheres The sphere of radius a centered at the origin has three common parametrizations:

1. As two graphs in Cartesian co-ordinates,

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} \pm \sqrt{a^2 - x^2 - y^2}\,\mathbf{k}, \quad -a \le x \le a,\ -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$$

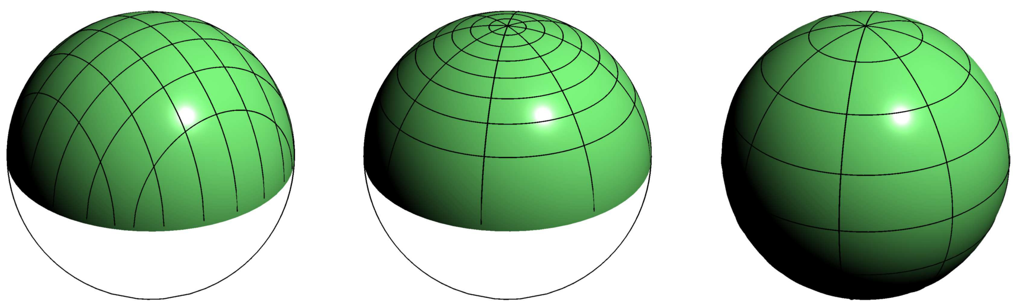
2. As two graphs in polar co-ordinates ($r = x\cos\theta$, $y = r\sin\theta$),

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} \pm \sqrt{a^2 - r^2}\,\mathbf{k}, \quad 0 \le r \le a,\ 0 \le \theta < 2\pi$$

3. Using spherical polar co-ordinates,

$$\mathbf{r}(\phi,\theta) = a\big(\sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}\big), \quad 0 \le \phi < \pi,\ 0 \le \theta \le 2\pi$$

The grid lines for each are drawn: in the first two cases we need the $\pm$ if we want both hemispheres!



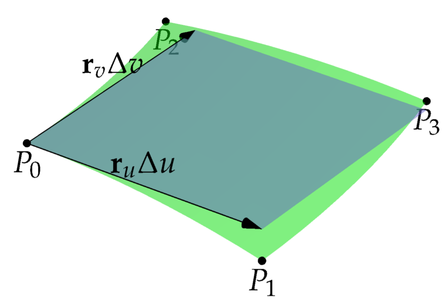
Surface Area of a Regular Surface

Suppose S has regular parametrization $\mathbf{r} : D \to \mathbb{R}^3$ and consider a point P_0 with position vector $\mathbf{r}(u_0, v_0)$. Increasing each co-ordinate by a small quantity $\Delta u, \Delta v$ produces three new points on the surface P_1, P_2, P_3, and describes a *parallelogram* in the tangent plane at P_0. This parallelogram has sides

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \mathbf{r}_u(u_0, v_0)\Delta u \qquad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \mathbf{r}_v(u_0, v_0)\Delta v$$

where the approximations become equality as $\Delta u, \Delta v \to 0$. The parallelogram approximates a small piece of surface bounded by the grid lines through P_0, P_1, P_2, P_3:

$$\Delta S = \{\mathbf{r}(u,v) : u_0 \le u \le u_0 + \Delta u, v_0 \le v \le v_0 + \Delta v\}$$



Since cross-products compute the area of a parallelogram, the area of this small piece of the surface is

$$A(\Delta S) \approx |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$$

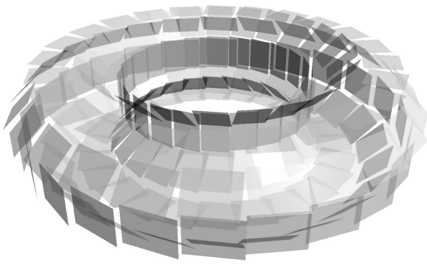
The surface area of S may therefore be approximated by a Riemann sum: taking the limit, we obtain…

Theorem. *If a surface S has a regular parametrization $\mathbf{r}(u,v)$ where (u,v) are co-ordinates on some domain $D \subseteq \mathbb{R}^2$, then the surface area of S is the double integral*

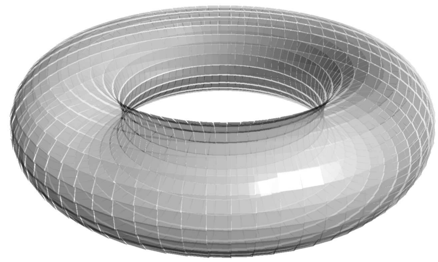
$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}u\,\mathrm{d}v$$

*Moreover, the **area element** $\mathrm{d}S := |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}u\,\mathrm{d}v$ is independent of regular parametrization.*

This may feel a little sketchy, but hopefully the following pictures help convince.



Torus approximated by 192 parallelograms



Torus approximated by 1728 parallelograms

The pictures also help explain why we don't just compute $\iint_D \mathrm{d}u\,\mathrm{d}v$: the $|\mathbf{r}_u \times \mathbf{r}_v|$ term corrects for the fact that parallelograms corresponding to the same $\Delta u, \Delta v$ are smaller on the "hole" part of the torus than the outside.

Surface Area of a Graph If $\mathbf{r}(u,v) = x\,\mathbf{i} + y\,\mathbf{j} + f(x,y)\,\mathbf{k}$ parametrizes the graph of $z = f(x,y)$, then its surface area is

$$A(S) = \iint_D \left|\mathbf{r}_x \times \mathbf{r}_y\right| \mathrm{d}x\,\mathrm{d}y = \iint_D \left| \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix} \right| \mathrm{d}x\,\mathrm{d}y = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,\mathrm{d}x\,\mathrm{d}y$$

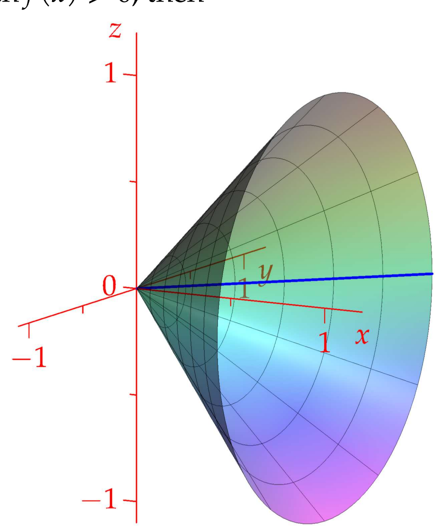
For the saddle surface $z = x^2 - y^2$ on the unit disk D,

$$A(S) = \iint_D \sqrt{4x^2 + 4y^2 + 1}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{6}(5\sqrt{5} - 1)$$

This integral should be evaluated by changing to polar co-ordinates, or instead we could have started with the polar parametrization $\tilde{\mathbf{r}}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r^2\cos 2\theta\,\mathbf{k}$. The result is the same either way.
As a sanity check, the surface area is approximately 170% that of the unit disk.

Surface of Revolution If $\mathbf{r}(x,\theta) = x\,\mathbf{i} + f(x)(\cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k})$ with $f(x) > 0$, then

$$\begin{aligned} A(s) &= \int_0^{2\pi} \int_{x_0}^{x_1} = \left| \begin{pmatrix} 1 \\ f'(x)\cos\theta \\ f'(x)\sin\theta \end{pmatrix} \times f(x) \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} \right| \mathrm{d}x\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \int_{x_0}^{ux_1} f(x)\sqrt{f'(x)^2 + 1}\,\mathrm{d}x\,\mathrm{d}\theta \\ &= 2\pi \int_{x_0}^{x_1} f(x)\sqrt{f'(x)^2 + 1}\,\mathrm{d}x \end{aligned}$$



For instance the surface area of the cone obtained by rotating the line $y = x$ around the x-axis when $0 \le x \le 1$ is

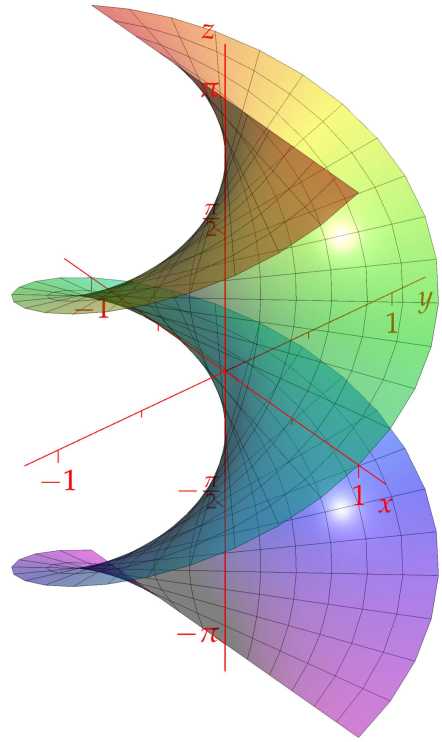
$$A = 2\pi \int_0^1 x\sqrt{1^2 + 1}\,\mathrm{d}x = \sqrt{2}\pi$$

Helicoid The parametrization

$$\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$$

with $-1 \le u \le 1$ and $-\pi \le v \le \pi$, describes the surface obtained by moving a horizontal line upwards while rotating it around the z-axis. Its surface area is

$$\begin{aligned}
A(S) &= \int_{-\pi}^{\pi} \int_{-1}^{1} \left| \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \times \begin{pmatrix} -u \sin v \\ u \cos v \\ 1 \end{pmatrix} \right| \mathrm{d}u\, \mathrm{d}v \\
&= \int_{-\pi}^{\pi} \int_{-1}^{1} \left| \begin{pmatrix} \sin v \\ -\cos v \\ u \end{pmatrix} \right| \mathrm{d}u\, \mathrm{d}v \\
&= \int_{-\pi}^{\pi} \int_{-1}^{1} \sqrt{1 + u^2}\, \mathrm{d}u\, \mathrm{d}v \\
&= 2\pi \big(u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2}) \big) \Big|_0^1 \\
&= 2\pi \big(\sqrt{2} + \ln(1 + \sqrt{2}) \big)
\end{aligned}$$

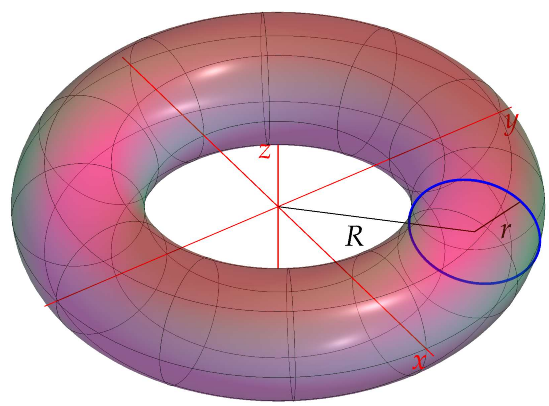


Torus Rotate the center of a small circle or radius r around the z-axis so that its center traces a circle of radius R. The resulting surface has parametric equation

$$\mathbf{r}(u, v) = \begin{pmatrix} (R + r\cos v)\cos u \\ (R + r\cos v)\sin u \\ r \sin v \end{pmatrix}$$

where $0 \le u, v \le 2\pi$. Since

$$\begin{aligned}
|\mathbf{r}_u \times \mathbf{r}_v| &= \left| \begin{pmatrix} -(R + r\cos v)\sin u \\ (R + r\cos v)\cos u \\ 0 \end{pmatrix} \times \begin{pmatrix} -r \sin v \cos u \\ -r \sin v \sin u \\ r \cos v \end{pmatrix} \right| \\
&= r(R + r\cos v)
\end{aligned}$$



the surface area of the torus is

$$\int_0^{2\pi} \int_0^{2\pi} r(R + r\cos v)\, \mathrm{d}u\, \mathrm{d}v = 4\pi^2 rR$$