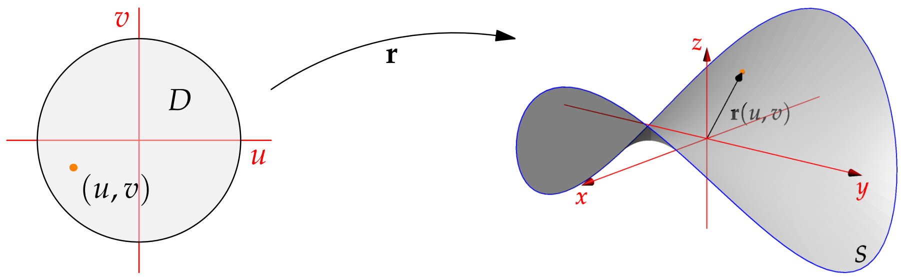
## 16.6 Parametric Surfaces and their Areas

Can define surfaces similarly to spacecurves: need two parameters $u, v$ instead of just $t$.

**Definition.** *Let $x, y, z$ be functions of two variables $u, v$, all with the same domain $D$. The* parametric surface *defined by the* co-ordinate functions *$x, y, z$ is the collection $S$ of position vectors*

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad \textit{for all } (u,v) \in D$$



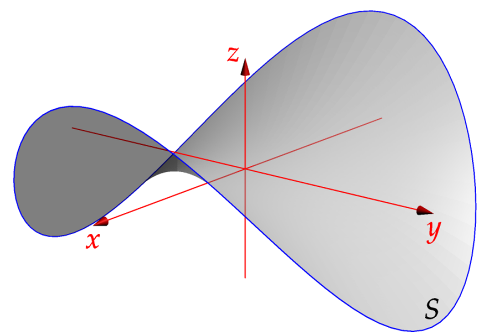

We think of a parameterized surface as a function $\mathbf{r} : D \to \mathbb{R}^3$.

**Example** The surface $S$ parameterized by

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\cos 2v\mathbf{k}$$

is drawn for $0 \le u \le 1$ and $0 \le v < 2\pi$. Since

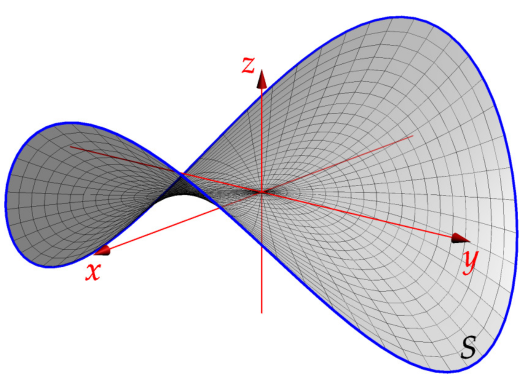
$$\begin{aligned} x^2 - y^2 &= u^2\cos^2 v - u^2\sin^2 v \\ &= u^2\cos 2v = z \end{aligned}$$



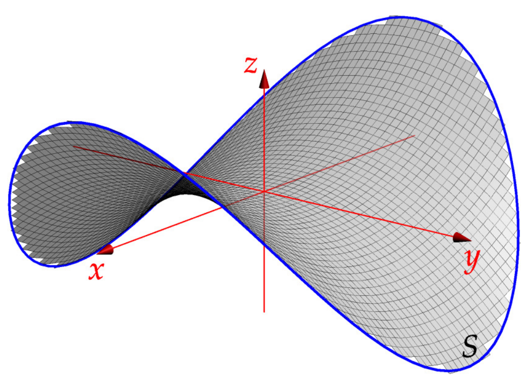

we recognize $S$ as the saddle surface $z = x^2 - y^2$

**Definition.** *The* grid lines *of a parametric surface $S$ are the two families of curves for which either $u$ or $v$ is constant.*

Important: *surface* distinct from *parameterization!* All surfaces can be parameterized in *infinite many ways.* Below are two different parameterizations of the saddle surface with the grid lines marked. Note how the grid lines look like a map for the surface. If you rotate the images to look at the $(x, y)$-plane, the grid lines should look familiar.




$$\mathbf{r}_1(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ u^2\cos 2v \end{pmatrix}$$




$$\mathbf{r}_2(u,v) = \begin{pmatrix} u \\ v \\ u^2 - v^2 \end{pmatrix}$$

**Standard Parameterized Surfaces**

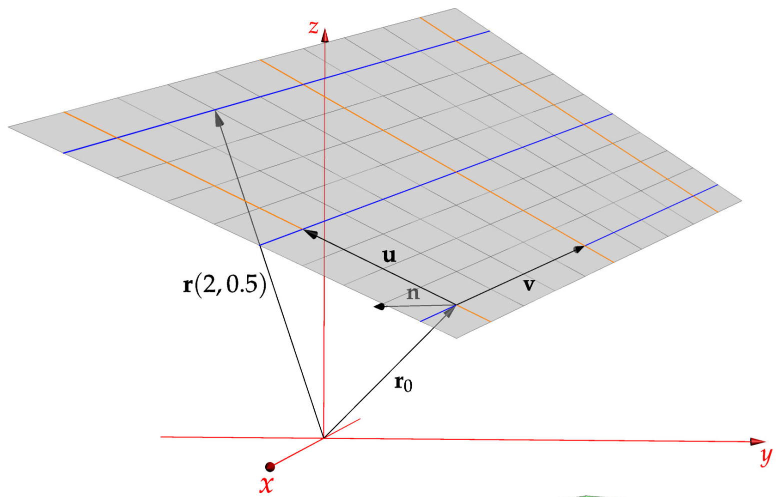
**Planes** The plane through a point with position vector $\mathbf{r}_0$ and containing the fixed vectors $\mathbf{u}$ and $\mathbf{v}$ has parametric equation

$$\mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{u} + v\mathbf{v}, \quad u,v \in \mathbb{R}$$

The grid lines are parallel to $\mathbf{u}, \mathbf{v}$. Equivalent to $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ where $\mathbf{n} = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$ is the unit normal vector.

The blue lines in the picture are the grid lines with $u = 0$, $u = 1$ and $u = 2$ respectively.
The orange lines are $v = 0$, $v = 1$ and $v = 2$.

Any other choice of $\mathbf{r}_0, \mathbf{u}, \mathbf{v}$ leads to a new parameterization.



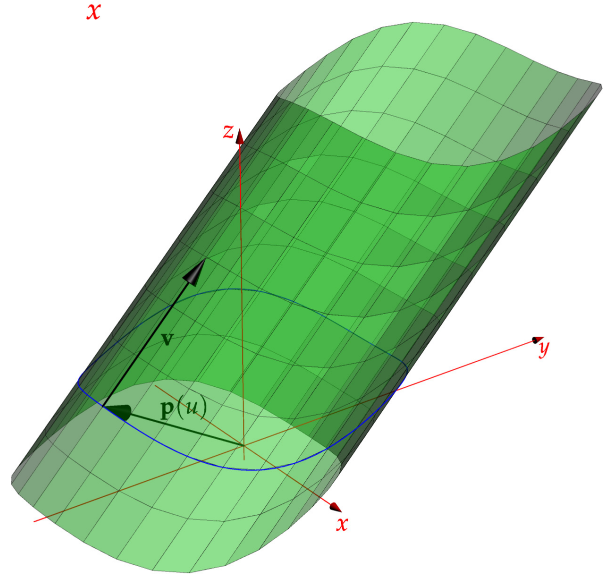

**Cylinders** A *cylinder* is a parametric surface

$$\mathbf{r}(u,v) = \mathbf{p}(u) + v\mathbf{v}, \quad u_0 \le u \le u_1, \quad v \in \mathbb{R},$$

where $\mathbf{p}(u)$ is a spacecurve and $\mathbf{v}$ is a constant vector.

The grid lines come in two families: copies of the original spacecurve translated in the $\mathbf{v}$-direction, and lines parallel to $\mathbf{v}$.

A *right-circular cylinder* is a cylinder where $\mathbf{p}(u)$ is a circle and $\mathbf{v}$ is perpendicular to the plane of the circle.




**Graphs of Functions** The graph of any function $z = f(x,y)$ is automatically parameterized: take $(u,v) = (x,y)$ to obtain

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + f(u,v)\mathbf{k}$$

We can also consider graphs parameterized using polar-co-ordinates $(u,v) = (r,\theta)$

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + f(u,v)\mathbf{k}$$

The saddle surface $z = x^2 - y^2$ was parameterized earlier using both these methods.

It is often easiest to think of a surface as a graph: for example...

Parameterize the surface $S$, being the part of the paraboloid $z = 10 - x^2 - y^2$ lying inside the cylinder $(x-1)^2 + y^2 = 4$.

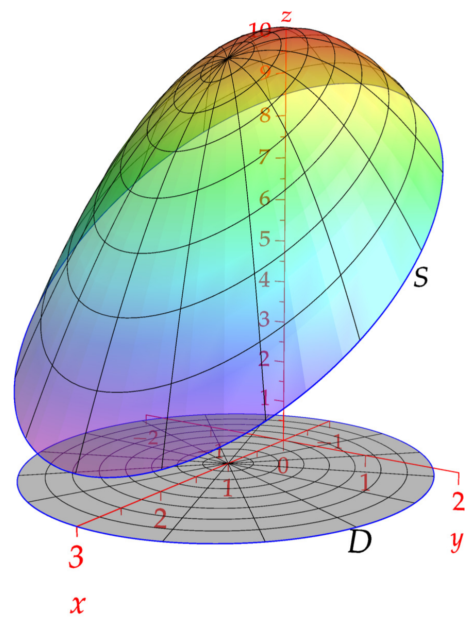
Viewing $S$ as a graph, we first project onto the $xy$-plane to obtain the domain of the function: the disk $D$ of radius 2 centered at $(1, 0)$.
Now parameterize $D$ using modified polar co-ordinates:

$$(x, y) = (1 + u\cos v, u \sin v)$$

Finally, compute $z$ using the formula for the paraboloid to obtain the parameterization

$$\mathbf{r}(u, v) = \begin{pmatrix} 1 + u\cos v \\ u\sin v \\ 10 - (1 + u\cos v)^2 - u^2\sin^2 v \end{pmatrix} = \begin{pmatrix} 1 + u\cos v \\ u\sin v \\ 9 - u^2 - 2u\cos v \end{pmatrix}$$

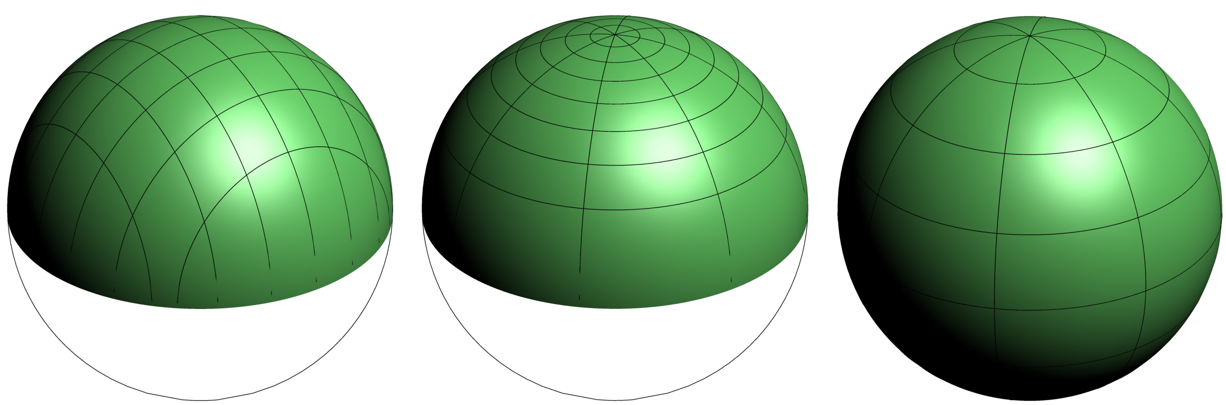
where $0 \le u \le 2$, $0 \le v < 2\pi$.




**Spheres** The sphere of radius $R$ has three common parameterizations:

- As two graphs in Cartesian co-ordinates: $z = \pm f(x, y) = \pm\sqrt{R^2 - x^2 - y^2}$.
- As two graphs in polar co-ordinates: $z = \pm f(r, \theta) = \pm\sqrt{R^2 - r^2}$.
- Using spherical co-ordinates: $(u, v) = (\phi, \theta) \implies \mathbf{r}(u, v) = R\begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}$

The grid lines for each are drawn below.



**Surfaces of revolution** Rotate a curve $y = f(x)$ for $a \le x \le b$ around the $x$-axis.

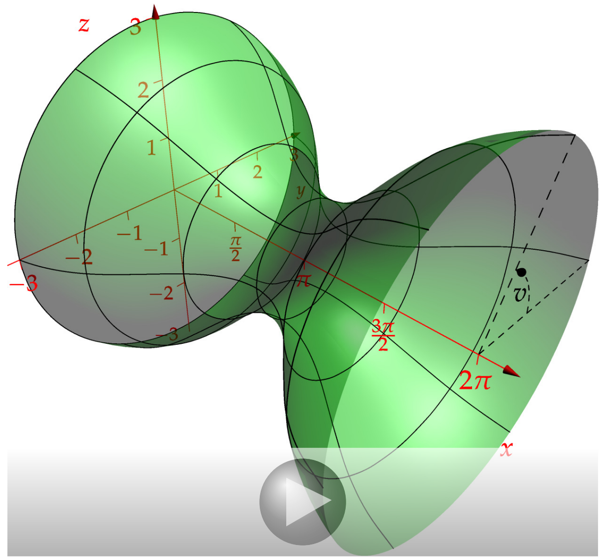
Letting $u$ measure the distance along the $x$-axis and $v$ the polar angle in the $yz$-plane, we obtain

$$\mathbf{r}(u,v) = u\mathbf{i} + f(u)(\cos v\mathbf{j} + \sin v\mathbf{k})$$

The plot is the result when $f(x) = 2 + \cos x$

The grid lines come in two families: copies of the original curve rotated around the $x$-axis, and circles centered on the $x$-axis.

Curves can be rotated around any axis for alternative versions.




**Tangent Planes**

**Definition.** *Let* $\mathbf{r}(u,v)$ *be a parameterization of S. A point* $\mathbf{r}(\hat{u},\hat{v})$ *is* regular *if* $(\mathbf{r}_u \times \mathbf{r}_v)(\hat{u},\hat{v}) \neq \mathbf{0}$.
*The* tangent plane *at a regular point* $\mathbf{r}(\hat{u},\hat{v})$ *is the plane through said point spanned by the* tangent vectors $\mathbf{r}_u(\hat{u},\hat{v})$ *and* $\mathbf{r}_v(\hat{u},\hat{v})$.
*The* unit normal vector *is* $\mathbf{n}(\hat{u},\hat{v}) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}(\hat{u},\hat{v})$.

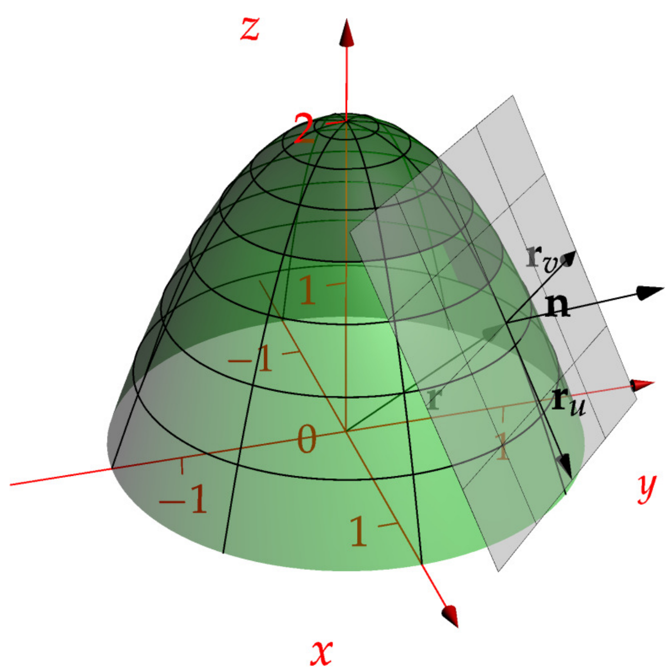
At $\mathbf{r}(1,\frac{\pi}{4}) = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1)$, the paraboloid

$$\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (2-u^2)\mathbf{k}$$

has tangent vectors

$$\mathbf{r}_u = \begin{pmatrix}\cos v\\ \sin v\\ -2u\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ -2\end{pmatrix}$$

$$\mathbf{r}_v = u\begin{pmatrix}-\sin v\\ \cos v\\ 0\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$$




The unit normal vector can be easily found:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{pmatrix}\sqrt{2}\\ \sqrt{2}\\ 1\end{pmatrix} \implies \mathbf{n} = \frac{1}{\sqrt{5}}\begin{pmatrix}\sqrt{2}\\ \sqrt{2}\\ 1\end{pmatrix}$$

The tangent plane therefore has equation

$$\sqrt{2}(x - \tfrac{1}{\sqrt{2}}) + \sqrt{2}(y - \frac{1}{\sqrt{2}}) + (z-1) = 0 \iff \sqrt{2}x + \sqrt{2}y + z = 3$$

**Surface area**

Let $S$ be parameterized by $\mathbf{r} : D \to \mathbb{R}^3$ and consider a point $P_0 = \mathbf{r}(a, b)$ on $S$. Increase each of the co-ordinates $u, v$ by small quantities $\Delta u, \Delta v$ to give three new points

$$P_1 = \mathbf{r}(a + \Delta u, b), \qquad P_2 = \mathbf{r}(a + \Delta u, b + \Delta v), \qquad P_3 = \mathbf{r}(a, b + \Delta v).$$

These four points are the corners of a small piece of surface

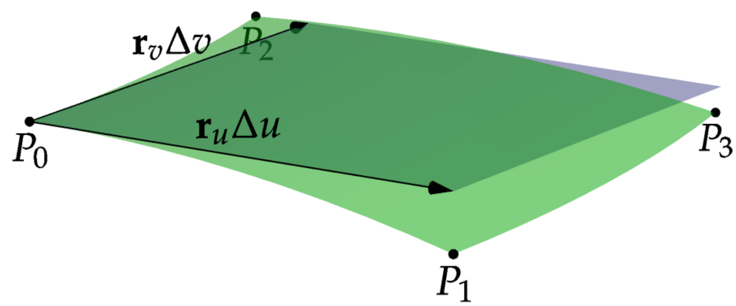
$$\Delta S = \{\mathbf{r}(u, v) : a \le u \le a + \Delta u,\ b \le v \le b + \Delta v\}$$

By the definition of partial derivative, we see that

$$\frac{\mathbf{r}(a + \Delta u, b) - \mathbf{r}(a, b)}{\Delta u} \approx \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}(a, b) \implies \mathbf{r}(a + \Delta u, b) - \mathbf{r}(a, b) \approx \mathbf{r}_u(a, b)\Delta u$$

Similarly $\mathbf{r}(a, b + \Delta v) - \mathbf{r}(a, b) \approx \mathbf{r}_v(a, b)\Delta v$.

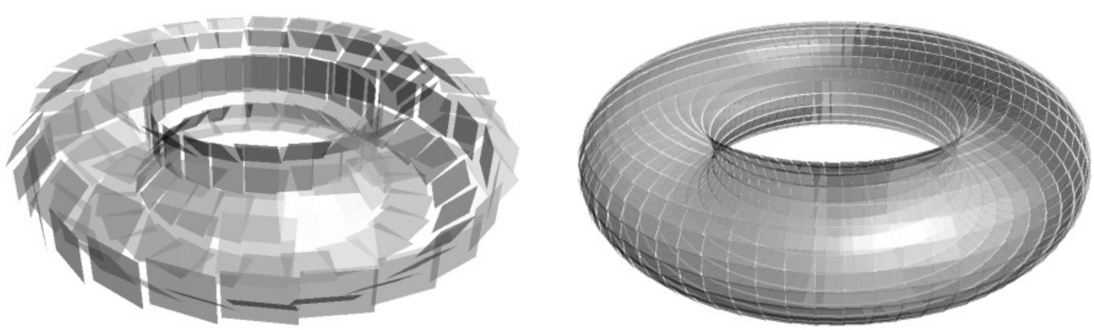
It follows that the small piece of surface $\Delta S$ is approximately the parallelogram in the tangent plane at $P_0$ with sides $\mathbf{r}_u\Delta u$ and $\mathbf{r}_v\Delta v$. Since cross-products compute the area of a parallelogram, we see that the area of $\Delta S$ is




$$A(\Delta S) \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u \Delta v$$

We can approximate the surface area of the entire surface $S$ by summing the areas of small approximating parallelograms. Explicitly, if the domain $D$ is a rectangle, subdivided into $mn$ subrectangles, each containing a sample point $(u_i, v_j)$,, then the area of $S$ is approximately the Riemann sum

$$\text{Area}(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \left|\mathbf{r}_u(u_i, v_j) \times \mathbf{r}_v(u_i, v_j)\right| \Delta u \Delta v$$



Torus approximated by 192 and 1728 parallelograms resp.

Taking limits as $m, n \to \infty$, we obtain…

**Theorem.** *If a surface $S$ is defined by the vector-valued function $\mathbf{r}(u, v)$ where $u, v$ are co-ordinates on some domain $D \subseteq \mathbb{R}^2$, then the surface area of $S$ is given by*

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}u\, \mathrm{d}v$$

**Definition.** $\mathrm{d}S = |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}u\, \mathrm{d}v$ *is the* area element *of a surface.*

**Lemma.** $\mathrm{d}S$ *is independent of co-ordinates. Indeed, if* $u = u(\alpha, \beta)$ *and* $v = v(\alpha, \beta)$ *for some alternative co-ordinates* $(\alpha, \beta)$*, then*

$$\left.\begin{array}{l} |\mathbf{r}_u \times \mathbf{r}_v| = \left|\frac{\partial(\alpha,\beta)}{\partial(u,v)}\right| |\mathbf{r}_\alpha \times \mathbf{r}_\beta| \\ \mathrm{d}u\,\mathrm{d}v = \left|\frac{\partial(u,v)}{\partial(\alpha,\beta)}\right| \mathrm{d}\alpha\,\mathrm{d}\beta \end{array}\right\} \implies |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}u\,\mathrm{d}v = \left|\mathbf{r}_\alpha \times \mathbf{r}_\beta\right| \mathrm{d}\alpha\,\mathrm{d}\beta$$

**Surface Area of a Graph** If $S$ is the graph of $z = f(x, y)$ for $(x, y) \in D$, then we may parameterize with $u = x$ and $v = y$:

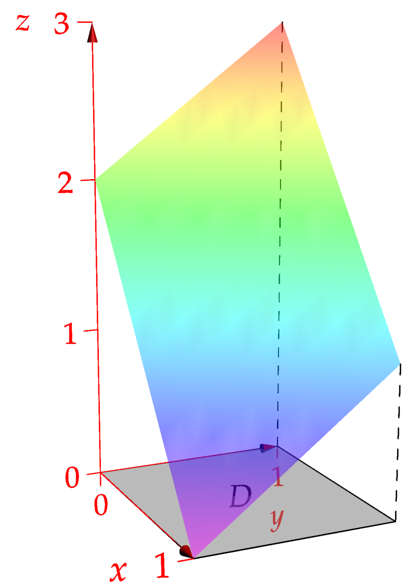
$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

Therefore

$$|\mathbf{r}_u \times \mathbf{r}_v| = \left|\mathbf{r}_x \times \mathbf{r}_y\right| = \left|\begin{pmatrix}1\\0\\f_x\end{pmatrix} \times \begin{pmatrix}0\\1\\f_y\end{pmatrix}\right| = \left|\begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix}\right|$$
$$= \sqrt{f_x^2 + f_y^2 + 1}$$

Hence the surface area is

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, \mathrm{d}x\, \mathrm{d}y$$




**Example** The plane $z = y - 2x + 2$ for $0 \le x, y \le 1$ has surface area

$$A(S) = \int_0^1 \int_0^1 \sqrt{(-2)^2 + 1^2 + 1}\, \mathrm{d}x\, \mathrm{d}y = \sqrt{6}$$

**Surfaces of Revolution** If $S$ is parameterized by $u = x$ and $v = \theta$, then

$$\mathbf{r}(u, v) = u\mathbf{i} + f(v)(\cos v\mathbf{j} + \sin v\mathbf{k})$$

Supposing that $a \le x \le b$ and $0 \le \theta < 2\pi$, we have

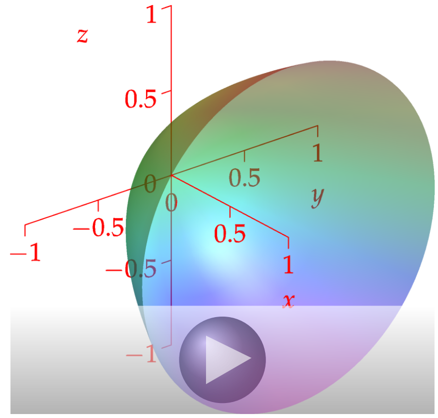
$$|\mathbf{r}_u \times \mathbf{r}_v| = |\mathbf{r}_x \times \mathbf{r}_\theta| \left|\begin{pmatrix}1\\ f'(x)\cos\theta \\ f'(x)\sin\theta\end{pmatrix} \times f(x)\begin{pmatrix}0\\ -\sin\theta \\ \cos\theta\end{pmatrix}\right| = \left|f(x)\begin{pmatrix}f'(x)\\ -\cos\theta \\ -\sin\theta\end{pmatrix}\right|$$
$$= f(x)\sqrt{f'(x)^2 + 1} \qquad \text{(supposing } f(x) \ge 0\text{)}$$

The surface area is therefore

$$A(S) = \int_0^{2\pi} \int_a^b f(x)\sqrt{f'(x)^2 + 1}\, \mathrm{d}x\, \mathrm{d}\theta = 2\pi \int_a^b f(x)\sqrt{f'(x)^2 + 1}\, \mathrm{d}x$$

**Example** If $f(x) = \sqrt{x}$ for $0 \le x \le 1$ then its surface area is

$$A = 2\pi \int_0^1 x^{1/2} \sqrt{\left(\frac{1}{2\sqrt{x}}\right)^2 + 1}\,\mathrm{d}x$$
$$= \pi \int_0^1 (4x+1)^{1/2}\,\mathrm{d}x = \frac{\pi}{6}(5\sqrt{5} - 1)$$



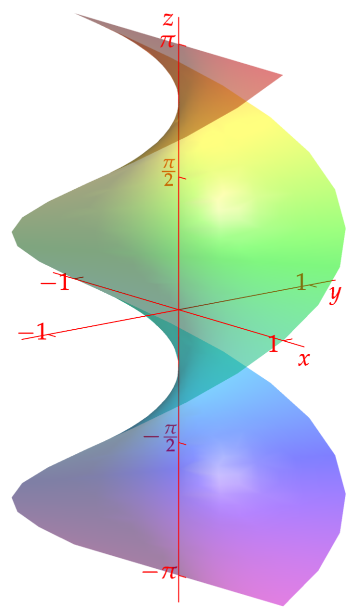

**More General Surfaces** Expect hard calculations where you have parameterize the surface and calculate $\mathbf{r}_u \times \mathbf{r}_v$ yourself! For example...

**A helicoid** is parameterized using polar co-ordinates $(u, v) = (r, \theta)$:

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$$

for $-1 \le u \le 1$ and $-\pi \le v \le \pi$. Its surface area is

$$A(S) = \int_{-\pi}^{\pi} \int_{-1}^{1} |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}u\,\mathrm{d}v$$
$$= \int_{-\pi}^{\pi} \int_{-1}^{1} \left| \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin v \\ u\cos v \\ 1 \end{pmatrix} \right| \mathrm{d}u\,\mathrm{d}v$$
$$= \int_{-\pi}^{\pi} \int_{-1}^{1} \left| \begin{pmatrix} \sin v \\ -\cos v \\ u \end{pmatrix} \right| \mathrm{d}u\,\mathrm{d}v$$
$$= \int_{-\pi}^{\pi} \int_{-1}^{1} \sqrt{1+u^2}\,\mathrm{d}u\,\mathrm{d}v$$
$$= 2\pi\left(u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2})\right)\Big|_0^1$$
$$= 2\pi(\sqrt{2} + \ln(1+\sqrt{2}))$$

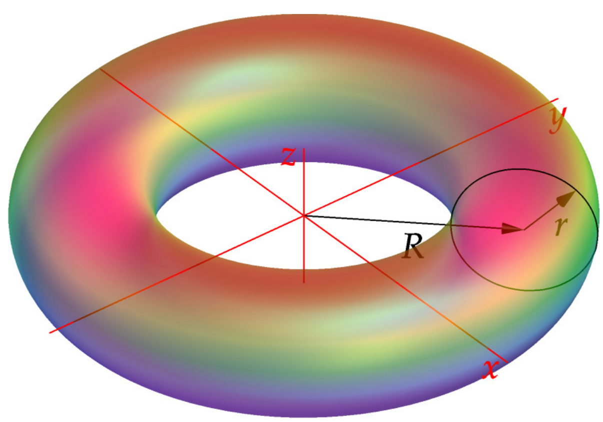



**A torus** has parametric equation

$$\mathbf{r}(u, v) = \begin{pmatrix} (R + r\cos v)\cos u \\ (R + r\cos v)\sin u \\ r\sin v \end{pmatrix}$$

where $0 \le u, v \le 2\pi$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \left| \begin{pmatrix} -(R + r\cos v)\sin u \\ (R + r\cos v)\cos u \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin v\cos u \\ -r\sin v\sin u \\ r\cos v \end{pmatrix} \right|$$
$$= r(R + r\cos v)$$




Hence the surface area is

$$\int_0^{2\pi} \int_0^{2\pi} r(R + r\cos v)\,\mathrm{d}u\,\mathrm{d}v = 4\pi^2 rR$$