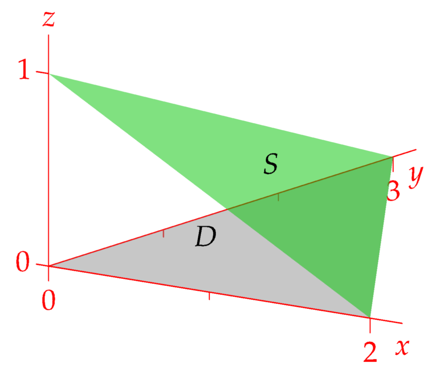
16.7 Surface Integrals

Definition. Suppose a surface S is parametrized by $\mathbf{r}(u,v)$ where $(u,v) \in D \subseteq \mathbb{R}^2$, and that $f : S \to \mathbb{R}$ is continuous on S. The integral of f over S is defined to be

$$\iint_S f \, \mathrm{d}S = \iint_D f(\mathbf{r}(u,v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$$

This is the same $\mathrm{d}S$ from previously, so any formulae from special cases can be utilized directly.

Example 1 Compute the integral of $f(x,y,z) = x^2$ over the part of the plane $3x + 2y + 6z = 6$ lying in the positive octant.



View S as the graph of the function $z = g(x,y) = 1 - \frac{1}{2}x - \frac{1}{3}y$ above the triangle

$$D : \quad 0 \le x \le 2; \quad 0 \le y \le 3 - \frac{3}{2}x$$

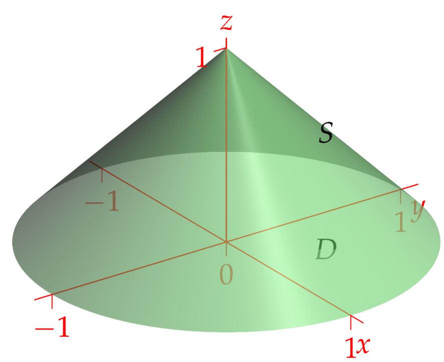
The area element using standard rectangular co-ordinates is therefore

$$\mathrm{d}S = \sqrt{g_x^2 + g_y^2 + 1} \, \mathrm{d}x \, \mathrm{d}x = \sqrt{\frac{1}{4} + \frac{1}{9} + 1} \, \mathrm{d}x \, \mathrm{d}y = \frac{7}{6} \, \mathrm{d}x \, \mathrm{d}y$$

and the required integral is

$$\iint_S x^2 \, \mathrm{d}S = \frac{7}{6} \int_0^2 \int_0^{3-\frac{3}{2}x} x^2 \, \mathrm{d}y \, \mathrm{d}x = \frac{7}{6} \int_0^2 3x^2 - \frac{3}{2}x^3 \, \mathrm{d}x = \frac{7}{3}$$

Example 2 Calculate the average height of a point on the surface of the cone $(1-z)^2 = x^2 + y^2$ lying above the x,y-plane.



We parametrize the surface using polar co-ordinates (r,θ) on the base disk in the (x,y)-plane:

$$\mathbf{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1-r \end{pmatrix}, \qquad 0 \le r \le 1, \quad 0 \le \theta < 2\pi$$

Therefore

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ -1 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix}$$

$$\implies \mathrm{d}S = \sqrt{2} r \, \mathrm{d}r \, \mathrm{d}\theta$$

The average height is therefore

$$z_{\text{av}} = \frac{\iint_S z \, \mathrm{d}S}{\iint_S \mathrm{d}S} = \frac{\int_0^{2\pi} \int_0^1 (1-r)\sqrt{2} r \, \mathrm{d}r \, \mathrm{d}\theta}{\int_0^{2\pi} \int_0^1 \sqrt{2} r \, \mathrm{d}r \, \mathrm{d}\theta} = \frac{1}{3}$$

Oriented Surfaces and Flux Integrals

To develop integrals of vector fields over surfaces, we need to describe which side of a surface is 'up.'

Definition. A connected surface S is *orientable* if a smooth choice of unit normal vector field $\mathbf{n}$ can be made: this choice is called an *orientation*.
Suppose $\mathbf{r}(u,v)$ parametrizes S. We say that (u,v) are *oriented co-ordinates* if $\mathbf{r}_u \times \mathbf{r}_v$ always points in the same direction as the orientation $\mathbf{n}$.

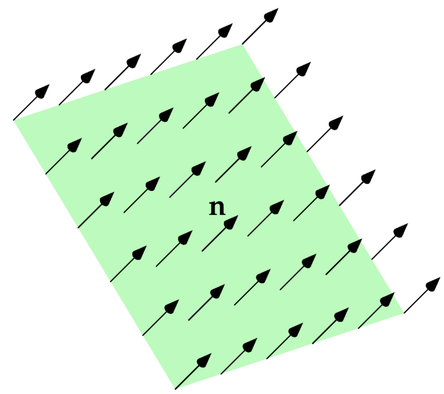
Loosely speaking, an orientable surface S has two faces. More precisely, it has exactly two orientations: reverse the normal field to obtain the other ($\mathbf{n} \mapsto -\mathbf{n}$).

There are several conventions when choosing an orientation.

1. Graphs of functions are typically have $\mathbf{n}$ pointing upwards. Equivalently, (x,y) or (r,θ) are oriented co-ordinates.

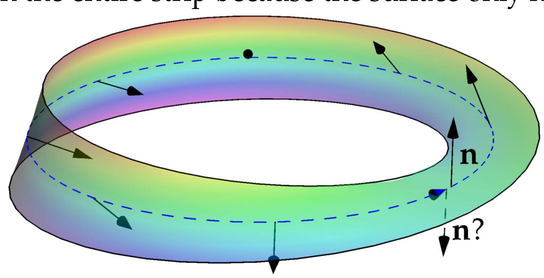
2. The *positive orientation* of a closed surface (such as a sphere) is when $\mathbf{n}$ points *outwards*.

3. If a surface is piecewise smooth, the orientation must be preserved across edges.



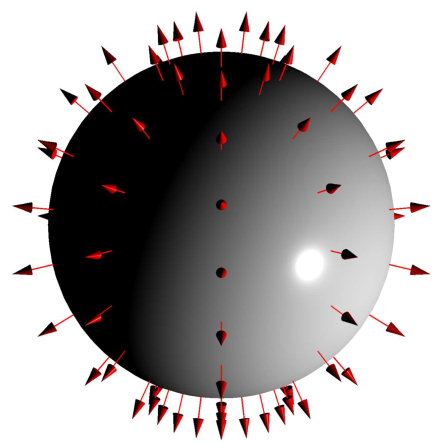
A choice of orientation of a plane

Not all surfaces are orientable. If we travel round the Möbius strip (below), the notion of 'up' reverses: it is impossible to smoothly choose $\mathbf{n}$ on the entire strip because the surface only has one face!



A Möbius strip (non-orientable)

The Klein Bottle is another famous non-orientable surface.



A positively oriented sphere

Definition. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector field $\mathbf{n}$, then the *flux integral* of $\mathbf{F}$ over S (or*surface integral*, or *flux* of $\mathbf{F}$ across S) is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where $\mathbf{n}\,\mathrm{d}S$ is the *vector area element* of the surface. If $(u,v) \in D$ are oriented co-ordinates, then

$$\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v = \mathbf{r}_u \times \mathbf{r}_v \, \mathrm{d}u \, \mathrm{d}v \implies \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, \mathrm{d}u \, \mathrm{d}v$$

If we reverse the orientation ($\mathbf{n} \mapsto -\mathbf{n}$), the flux changes sign.

You should compare all this with the 2D-flux encountered earlier: $\int_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) \, \mathrm{d}t$.

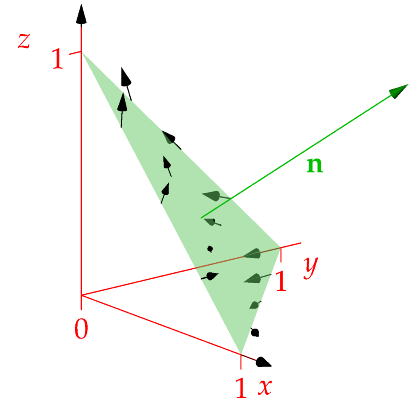
Example 1 Compare the fluxes of the vector fields $\mathbf{F}_1 = x^2\,\mathbf{i} - y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{F}_2 = x^2\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the upward-oriented plane $x + y + z = 1$ in the positive octant.

The plane is a graph, so we use rectangular co-ordinates:

$$\mathbf{r}(x,y) = \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix} \qquad 0 \le x \le 1, \quad 0 \le y \le 1-x$$

quickly gives the correct normal vector field $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (points upward!). The flux is therefore

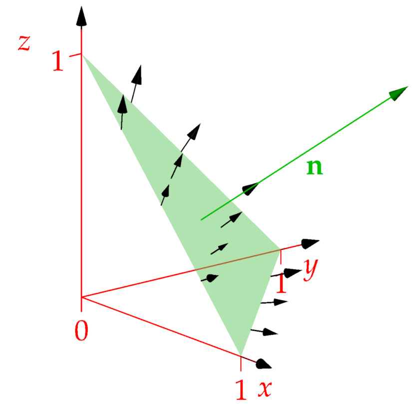
$$\iint_S \mathbf{F}_1 \cdot \mathrm{d}\mathbf{S} = \int_0^1 \int_0^{1-x} \begin{pmatrix} x^2 \\ -y \\ 1-x-y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} x^2 - x - 2y + 1\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{12}$$



Flux of $\mathbf{F}_1$ across S

For the second flux,

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^1 \int_0^{1-x} \begin{pmatrix} x^2 \\ y \\ 1-x-y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \int_0^{1-x} x^2 - x + 1\,\mathrm{d}y\,\mathrm{d}x = \frac{5}{12}$$

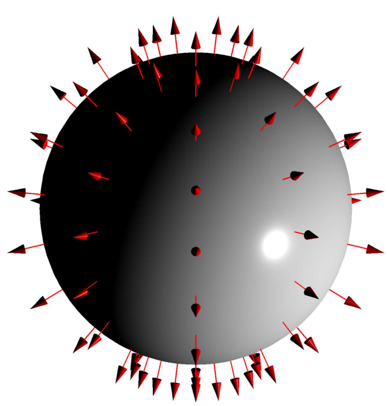


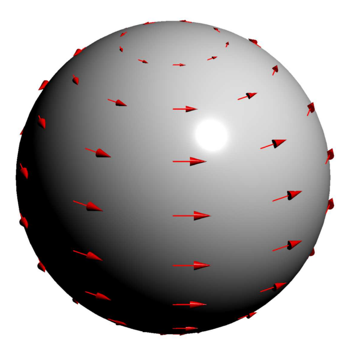
Flux of $\mathbf{F}_2$ across S

Consider how the directions of the arrows help explain the *fivefold* increase in flux. The second field generally has a larger component in the direction of the normal field $\mathbf{n}$, so the flux is larger.

Example 2 We compare the fluxes of the radial and rotational fields $\mathbf{F}_1 = \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{F}_2 = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ across the sphere of radius 1. Here $\mathbf{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r}$ (write $r = |\mathbf{r}|$) and so

$$\iint_S \mathbf{F}_1 \cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{r} \cdot \mathbf{r}\,\mathrm{d}S = \iint_S r^2\,\mathrm{d}S = \iint_S \mathrm{d}S = \mathrm{Area}(S) = 4\pi$$
$$\iint_S \mathbf{F}_2 \cdot \mathrm{d}\mathbf{S} = \iint_S \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mathrm{d}S = \iint_S \mathrm{d}S = 0$$

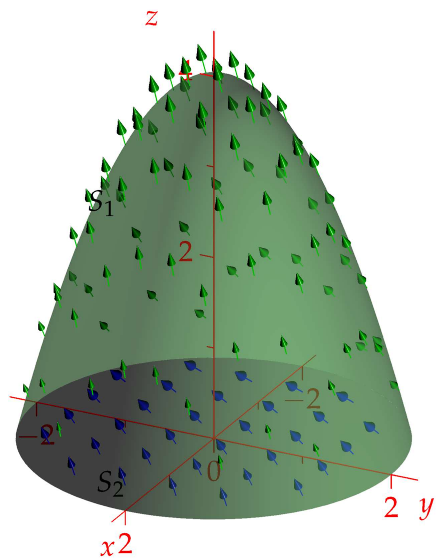

Arrows normal to S: flux > 0


Arrows tangent to S: flux $= 0$

Example 3: Net Flux We compute the flux of $\mathbf{F} = x\,\mathbf{i} + (5-z)\,\mathbf{k}$ out of S, which comprises two sub-surfaces:

S_1: the paraboloid $z = 4 - (x^2+y^2)$ for $z \geq 0$

S_2: the elliptical disk $x^2+y^2 \leq 4$ in the xy-plane



Parametrize S_1 using polar co-ordinates on the base:

$$\mathbf{r}_1(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 4-r^2 \end{pmatrix},\ 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi$$

we see that the normal vector field $\mathbf{n}$ points upwards. Indeed

$$\iint_{S_1} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^2 \begin{pmatrix} r\cos\theta \\ 0 \\ 1+r^2 \end{pmatrix} \cdot \begin{pmatrix} 2r^2\cos\theta \\ 2r^2\sin\theta \\ r \end{pmatrix} \mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^2 2r^3\cos^2\theta + r^3 + r\,\mathrm{d}r\,\mathrm{d}\theta = 20\pi$$

Over S_2, the outward pointing normal vector is $\mathbf{n} = -\mathbf{k}$ (*downward!*) and so

$$\iint_{S_2} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = -\iint_S \begin{pmatrix} x \\ 0 \\ 5-0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S = -5\iint_S \mathrm{d}S = -20\pi$$

since S_2 is a disk of radius 2. The net flux is therefore zero: there is as much flux into the region bounded by S as there is leaving it.[1]

Application: Heat Flow

If $T(x,y,z)$ measures temperature, the *heat flow* is the vector field $\mathbf{F} = -k\nabla T$, where k (the *thermal conductivity*) is a constant of the material. The *heat flux* out of a region with boundary surface S is

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = -k\iint_S \nabla T\cdot \mathbf{n}\,\mathrm{d}S$$

(Units: $[T] = \mathrm{K}$, $[\nabla T] = \mathrm{Km}^{-1}$, $[k] = \mathrm{Wm}^{-1}\mathrm{K}^{-1}$, $[\mathbf{F}] = \mathrm{Wm}^{-2}$)

Example At a given moment, a hot 10cm-radius sphere of gold ($k = 318\ \mathrm{Wm}^{-1}\mathrm{K}^{-1}$) has temperature $T(r) = \frac{500}{1+r}$ K, measured a distance r meters from its center. The heat flow is therefore

$$\mathbf{F} = -k\nabla T = \frac{159}{r(1+r)^2}\mathbf{r}\quad \mathrm{kWm}^{-2} \qquad (\nabla f(r) = \tfrac{f'(r)}{r}\mathbf{r})$$

Since the unit normal vector field on the surface is $\frac{1}{r}\mathbf{r}$, the total heat flux out of the sphere at the given moment is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{r=0.1} \frac{159}{(1+r)^2}\mathrm{d}S = \frac{159}{1.1^2}\iint_{r=0.1}\mathrm{d}S = \frac{159}{1.1^2}\cdot 4\pi(0.1)^2 = \frac{636\pi}{121} \approx 16.5\ \mathrm{kW}$$

[1]As we'll see later, the cheap reason for this is because the divergence of **F** is zero…