16.7 Surface Integrals

Recall: if D is a 2D-region, then $\iint_D f \, \mathrm{d}A = \text{Area}(D) \cdot f_{\text{av}}$

For surfaces: approximate S by parallelograms ΔS_{ij} located at points P_{ij}. On each parallelogram, $f_{\text{av}} \approx f(P_{ij})$, and so it seems sensible to have

$$\iint_S f \, \mathrm{d}S \approx \sum_{i,j} f(P_{ij}) \cdot \text{Area}(\Delta S_{ij})$$

where the approximation improves with more parallelograms.
If S is parameterized then we know how to compute the areas:

Theorem. *Suppose the surface S is parameterized by $\mathbf{r}(u,v)$ where $(u,v) \in D \subseteq \mathbb{R}^2$, and that f is continuous on S. The integral of f over S is defined to be*

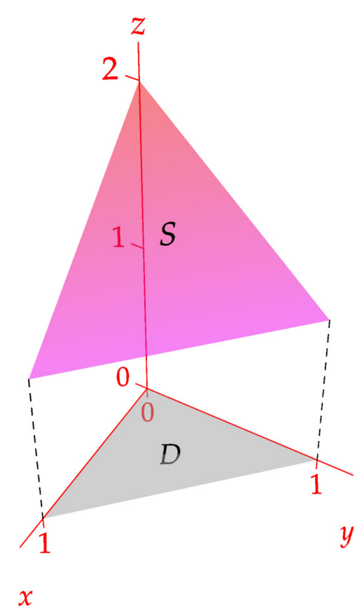
$$\iint_S f \, \mathrm{d}S = \iint_D f(\mathbf{r}(u,v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$$

Similar Notations Observe how the notation is similar to that for a line integral:

Surface Integral: $\displaystyle\iint_S f \, \mathrm{d}S = \iint_D f(\mathbf{r}(u,v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$

Line Integral: $\displaystyle\int_C f \, \mathrm{d}s = \int_a^b f(\mathbf{r}(t)) \, |\mathbf{r}'(t)| \, \mathrm{d}t$

Example Find the integral of the function $f(x,y,z) = x^2$ over the part of the part of the plane $x + y + z = 2$ above the triangle with corners $(0,0)$, $(1,0)$, $(0,1)$



The surface S is the graph of

$$z = g(x,y) = 2 - x - y$$

hence the area element is

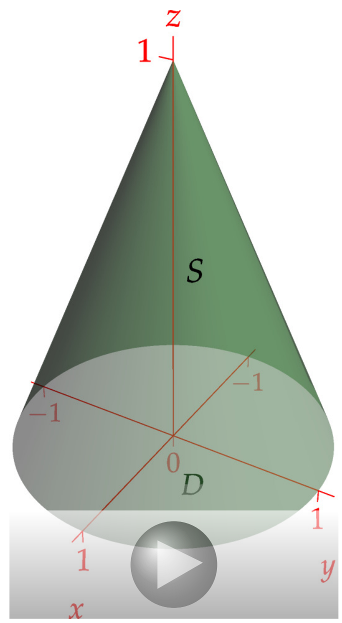
$$\mathrm{d}S = \sqrt{g_x^2 + g_y^2 + 1} \, \mathrm{d}x \, \mathrm{d}y = \sqrt{3} \, \mathrm{d}x \, \mathrm{d}y$$

The integral is therefore

$$\iint_S x^2 \, \mathrm{d}S = \iint_D x^2 \sqrt{3} \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_0^{1-x} x^2 \sqrt{3} \, \mathrm{d}y \, \mathrm{d}x$$
$$= \sqrt{3} \int_0^1 x^2 - x^3 \, \mathrm{d}x = \frac{\sqrt{3}}{12}$$

This example is easy because S is a *graph* and we can therefore see the domain D as part of the xy-plane. The parameterization $\mathbf{r}(u,v) = \begin{pmatrix} u \\ v \\ 2-u-v \end{pmatrix}$ and the computation of $\mathrm{d}S = |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$ does not need to be made explicit.

Example Calculate $\iint_S z\,\mathrm{d}S$ where S is the region of the cone $z = 1 - r$ in cylindrical polar co-ordinates lying above the x, y-plane



This is also a graph, although it is easiest to parameterize the cone using polar co-ordinates $(u, v) = (r, \theta)$:

$$\mathbf{r}(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ 1-u \end{pmatrix}, \qquad 0 \le u \le 1, \quad 0 \le v < 2\pi$$

Therefore

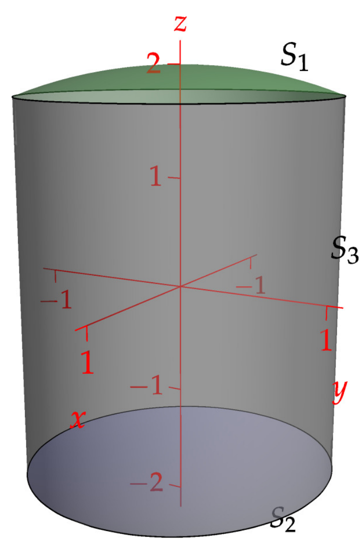
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{pmatrix} \cos v \\ \sin v \\ -1 \end{pmatrix} \times \begin{pmatrix} -u\sin v \\ u\cos v \\ 0 \end{pmatrix} = \begin{pmatrix} u\cos v \\ u\sin v \\ u \end{pmatrix}$$

$$\implies \mathrm{d}S = \sqrt{2}\,u\,\mathrm{d}u\,\mathrm{d}v$$

The integral is therefore

$$\iint_S z\,\mathrm{d}S = \iint_D (1-u)\sqrt{2}\,u\,\mathrm{d}u\,\mathrm{d}v = \sqrt{2}\int_0^{2\pi}\int_0^1 u - u^2\,\mathrm{d}u\,\mathrm{d}v = \frac{\sqrt{2}\pi}{3}$$

Piecewise Surfaces Surfaces can consist of multiple parts: we must compute a separate surface integral, with separate parameterization, for each piece.



For example, we compute the integral $\iint_S y^2 - z\,\mathrm{d}S$ where S comprises parts of the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$ as shown. The intersections happen at $z = \pm\sqrt{3}$

Take S_1 to be the upper part of the sphere, S_2 the lower, and S_3 the cylinder

There are several choices for how to parameterize each surface. We will use spherical co-ordinates for the parts of the spheres, and cylindrical co-ordinates for the cylinder.

It is straightforward to check that the maximum angle ϕ for the upper cap S_1 is $\phi = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. We therefore have the parameterizations:

$$\mathbf{r}_1(u,v) = \begin{pmatrix} 2\sin u\cos v \\ 2\sin u\sin v \\ 2\cos u \end{pmatrix} \qquad 0 \le u \le \frac{\pi}{6}, \quad 0 \le v < 2\pi \qquad ((u,v) = (\phi,\theta))$$

$$\mathbf{r}_2(u,v) = \mathbf{r}_1(u,v) \qquad \frac{5\pi}{6} \le u \le \pi, \quad 0 \le v < 2\pi$$

$$\mathbf{r}_3(u,v) = \begin{pmatrix} \cos u \\ \sin u \\ v \end{pmatrix} \qquad 0 \le u < 2\pi, \quad -\sqrt{3} \le v \le \sqrt{3} \qquad ((u,v) = (\theta,z))$$

Calculate the area elements:

$$\mathrm{d}S_1 = \mathrm{d}S_2 = \left| \begin{pmatrix} 2\cos u \cos v \\ 2\cos u \sin v \\ -2\sin u \end{pmatrix} \times \begin{pmatrix} -2\sin u \sin v \\ 2\sin u \cos v \\ 0 \end{pmatrix} \right| \mathrm{d}u\,\mathrm{d}v = 4\sin u \left| \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \right| \mathrm{d}u\,\mathrm{d}v$$
$$= 4\sin u\,\mathrm{d}u\,\mathrm{d}v$$

$$\mathrm{d}S_3 = \left| \begin{pmatrix} -\sin u \\ \cos u \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| \mathrm{d}u\,\mathrm{d}v = \left| \begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix} \right| \mathrm{d}u\,\mathrm{d}v = \mathrm{d}u\,\mathrm{d}v$$

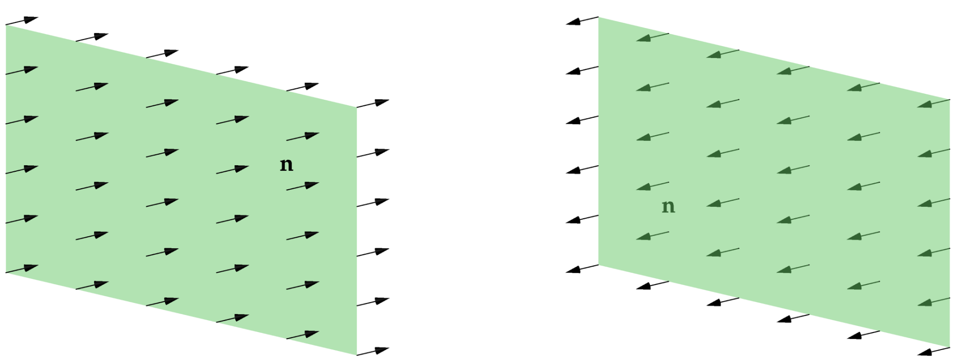
To compute $\iint_S z^2\,\mathrm{d}S$ we sum the three integrals:

$$\begin{aligned}
\iint_S z^2\,\mathrm{d}S &= \iint_{S_1} z^2\,\mathrm{d}S_1 + \iint_{S_2} z^2\,\mathrm{d}S_2 + \iint_{S_3} z^2\,\mathrm{d}S_2 \\
&= \left(\int_0^{2\pi}\int_0^{\frac{\pi}{6}} + \int_0^{2\pi}\int_{\frac{5\pi}{6}}^{\pi} \right) (2\cos u)^2 \cdot 4\sin u\,\mathrm{d}u\,\mathrm{d}v + \int_{-\sqrt{3}}^{\sqrt{3}}\int_0^{2\pi} v^2\,\mathrm{d}u\,\mathrm{d}v \\
&= 2\pi \left(\int_0^{\frac{\pi}{6}} + \int_{\frac{5\pi}{6}}^{\pi} \right) 16\cos^2 u \sin u\,\mathrm{d}u + 2\pi \int_{-\sqrt{3}}^{\sqrt{3}} v^2\,\mathrm{d}v \\
&= \frac{32\pi}{3}\left[-\cos^3 u\right]_0^{\frac{\pi}{6}} + \frac{32\pi}{3}\left[-\cos^3 u\right]_{\frac{5\pi}{6}}^{\pi} + \frac{4\pi}{3} v^3 \Big|_0^{\sqrt{3}} \\
&= \frac{32\pi}{3}\left(-(\tfrac{\sqrt{3}}{2})^3 + 1 - (-1) + (-\tfrac{\sqrt{3}}{2})^3\right) + 4\pi\sqrt{3} = \frac{4(16-3\sqrt{3})}{3}\pi
\end{aligned}$$

Orientation

An inhabitant of a surface needs to know which way is 'up'.

Definition. *A connected surface S is* orientable *if it has exactly two faces.*
An orientation *is a smooth choice of unit normal vector field*[1] $\mathbf{n}$.



Suppose that $\mathbf{r}(u,v)$ parameterizes S. We say that (u,v) are oriented co-ordinates *if $\mathbf{r}_u \times \mathbf{r}_v$ always points in the same direction as the orientation $\mathbf{n}$.*

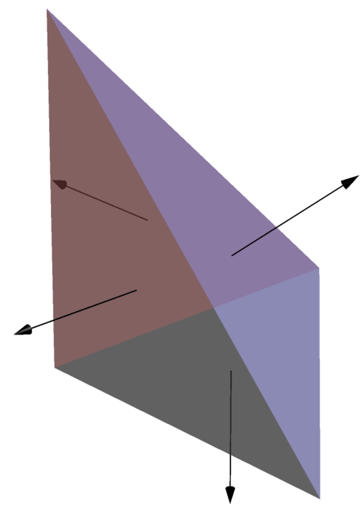
[1] At each point we have a choice of two unit normal vectors $\mathbf{n}$. An orientable surface therefore has *two* opposite possible orientations.

Choice of Orientation For most orientable surfaces the choice is entirely arbitrary, although there are several conventions.

1. Graphs of functions $z = f(x, y)$ or $z = f(r, \theta)$ are typically orientated so that the normal vectors point upwards. This is equivalent to having (x, y) and (r, θ) be oriented co-ordinates.

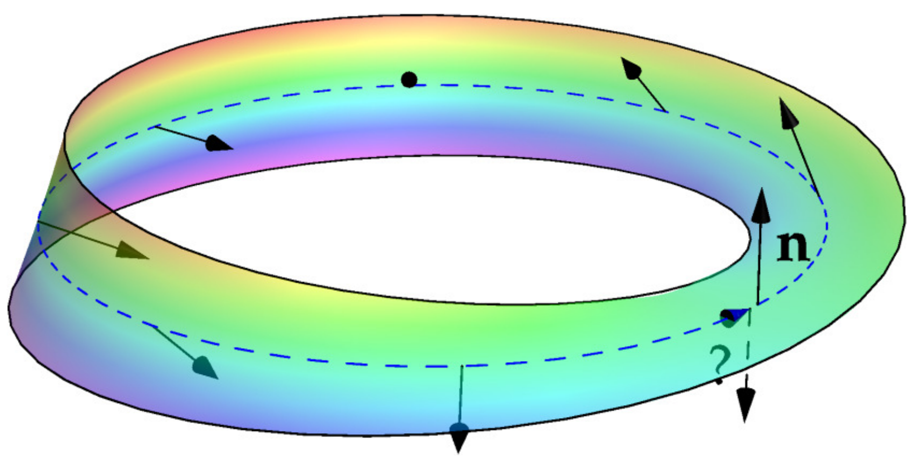
2. Closed surfaces (such as spheres) tend to be oriented so that $\mathbf{n}$ points outwards. We refer to this as *positive orientation* of a closed surface.

3. If a surface is piecewise smooth, the orientation *must* be preserved across edges.



A positively oriented tetrahedron is drawn. Notice that the orientation is preserved across edges, so that all faces are oriented outwards.

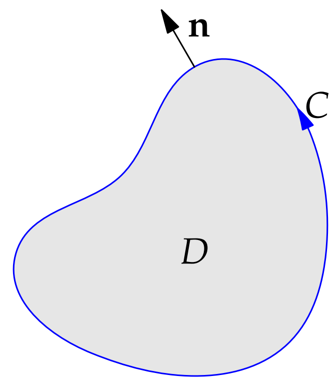
A non-orientable surface Not all surfaces are orientable, the Möbius strip being the classic example.[2]



Traveling round the dashed curve changes the notion of 'up': it is impossible to smoothly choose $\mathbf{n}$ on the entire surface because the surface only has one face.
Small parts of a Möbius strip can be oriented, but not the entire surface.

[2]The `Klein Bottle` is even weirder...

Flux Integrals



Recall an earlier physical interpretation of Green's Theorem.

If C is the boundary of a simply-connected region D, then $\mathbf{F} \cdot \mathbf{n}$ is the component of $\mathbf{F}$ pointing out of D: indeed

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \textit{total flow of } \mathbf{F} \text{ out of } D$$

This notion can be extended to 3-dimensions.

Definition. *If* $\mathbf{F}$ *is a continuous vector field defined on an oriented surface* S *with normal vector field* $\mathbf{n}$*, then the* flux integral[3] *of* $\mathbf{F}$ *over* S *is defined as*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$$

Notation If $(u, v) \in D$ are oriented co-ordinates on S, then $\mathbf{r}_u \times \mathbf{r}_v$ points in the same direction as $\mathbf{n}$, whence

$$\mathbf{n} \, \mathrm{d}S = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v = \mathbf{r}_u \times \mathbf{r}_v \, \mathrm{d}u \, \mathrm{d}v$$

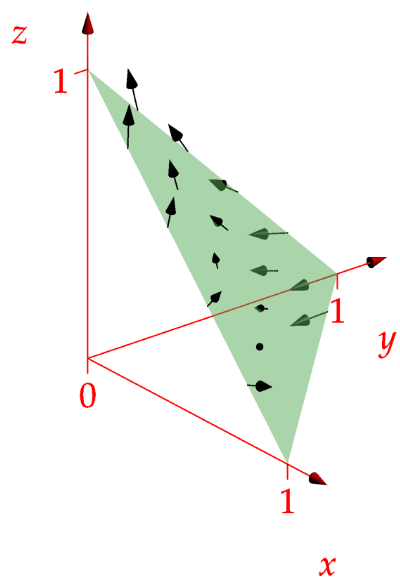
Because of this, it is conventional to define the *vector area element*[4]

$$\mathrm{d}\mathbf{S} = \mathbf{n} \, \mathrm{d}S = \mathbf{r}_u \times \mathbf{r}_v \, \mathrm{d}u \, \mathrm{d}v$$

and write the flux integral of $\mathbf{F}$ as

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, \mathrm{d}u \, \mathrm{d}v$$

Example Compute the flux of the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ across the upward-oriented plane $x + y + z = 1$ in the positive octant.



The plane is a graph, whence we may use co-ordinates $(u, v) = (x, y)$. The parameterization by $\mathbf{r}(u, v) = \begin{pmatrix} u \\ v \\ 1-u-v \end{pmatrix}$ where $0 \le u \le 1$ and $0 \le v \le 1 - u$, quickly gives the correct normal vector field $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The flux is then

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_0^1 \int_0^{1-u} \begin{pmatrix} u^2 \\ -v \\ 1-u-v \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{d}v \, \mathrm{d}u \\ &= \int_0^1 \int_0^{1-u} u^2 - u - 2v + 1 \, \mathrm{d}v \, \mathrm{d}u = \frac{1}{12} \end{aligned}$$

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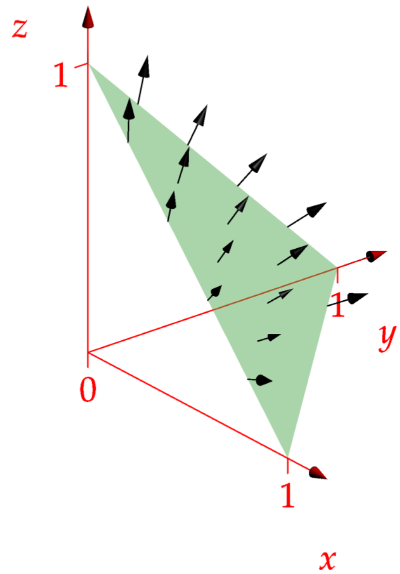
[3] Or *surface integral* or simply *flux*

[4] The distinction between $\mathrm{d}s = |\mathbf{r}'(t)| \, \mathrm{d}t$ and $\mathrm{d}\mathbf{r} = \mathbf{r}'(t) \, \mathrm{d}t$ is similar. The advantage of $\mathrm{d}\mathbf{S}$ and $\mathrm{d}\mathbf{r}$ are that they require no square-roots to compute.

Now we compute the same flux as before, but with a subtle change in the vector field $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^1 \int_0^{1-u} \begin{pmatrix} u^2 \\ v \\ 1-u-v \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \, dv\, du$$
$$= \int_0^1 \int_0^{1-u} u^2 - u + 1 \, dv\, du = \frac{5}{12}$$

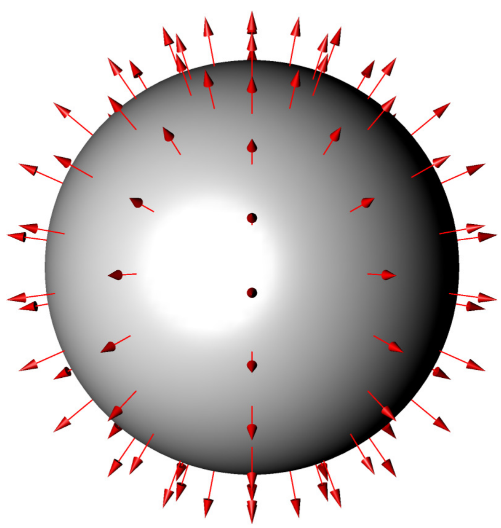
Note how the directions of the arrows explain the *fivefold* increase in flux. The vector field $\mathbf{F}$ is mostly pointing in the same direction as the normal vector $\mathbf{n}$, hence a positive flux.



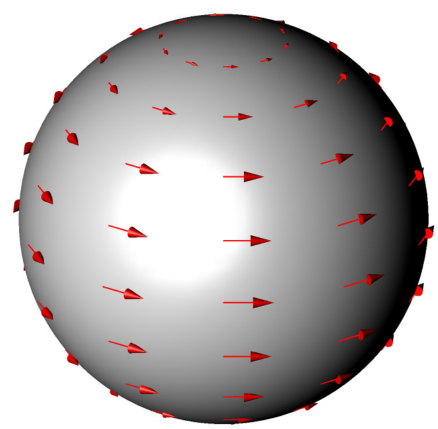
Example Compare the flux of the radial and rotational vector fields $\mathbf{F}_1 = \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{F}_2 = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ across the sphere of radius 1.

Here $\mathbf{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r}$ and so

$$\iint_S \mathbf{F}_1 \cdot d\mathbf{S} = \iint_S \mathbf{r}\cdot\mathbf{r}\, dS = \iint_S r^2\, dS = \iint_S dS = \text{Area}(S) = 4\pi$$
$$\iint_S \mathbf{F}_2 \cdot d\mathbf{S} = \iint_S \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \cdot \mathbf{r}\, dS = 0$$

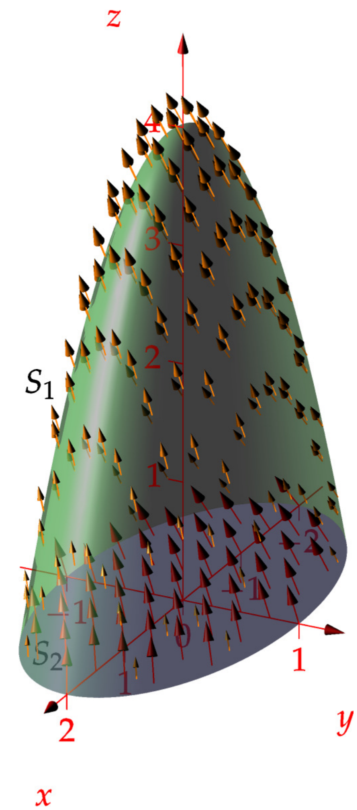


Arrows normal to S: flux > 0



Arrows tangent: flux $= 0$

Net Flux Flux can be both in and out of a region



For example, let S comprise two subsurfaces,
S_1: the paraboloid $z = 4 - x^2 - 4y^2$ for $z \geq 0$
S_2 the elliptical disk $x^2 + 4y^2 \leq 4$

Let $\mathbf{F} = (z - x)\mathbf{i} - \mathbf{j} + (z + 3)\mathbf{k}$

Parameterizing the paraboloid S_1 as a graph $(u, v) = (x, y)$ oriented upwards, we quickly compute the normal vector field $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{pmatrix} 2u \\ 8v \\ 1 \end{pmatrix}$, whence

$$\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D \begin{pmatrix} (4 - u^2 - 4v^2) - u \\ -1 \\ 4 - u^2 - 4v^2 + 3 \end{pmatrix} \cdot \begin{pmatrix} 2u \\ 8v \\ 1 \end{pmatrix} \mathrm{d}u\,\mathrm{d}v$$
$$= \iint_D (2u + 1)(4 - u^2 - 4v^2) - 2u^2 - 8v + 3\,\mathrm{d}u\,\mathrm{d}v$$

where D is the inside of the ellipse $u^2 + 4v^2 = 4$.

To compute this integral, change variables to modified polar co-ordinates $u = 2r\cos\theta$, $v = r\sin\theta$, so that $\mathrm{d}u\,\mathrm{d}v = 2r\,\mathrm{d}r\,\mathrm{d}\theta$. With a little substitution and cancelling, we obtain

$$\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \int_0^1 ((4r\cos\theta + 1)(4 - 4r^2) - 8r^2\cos^2\theta - 8r\sin\theta + 3)2r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_0^{2\pi} \int_0^1 ((4 - 4r^2 - 4r^2(1 + \cos 2\theta) + 3)2r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= 4\pi \int_0^1 (7 - 8r^2)r\,\mathrm{d}r = 6\pi$$

Over S_2, the outward pointing normal vector is $\mathbf{n} = -\mathbf{k}$ (*downward!*) and so

$$\iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -\iint_S \begin{pmatrix} -x \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S = -3 \iint_S \mathrm{d}S = -6\pi$$

since S_2 is an ellipse. The net flux is therefore zero: there is as much flux into the region bounded by S as there is leaving it.

Non-examinable Application: Heat Flow

If the temperature at a point (x, y, z) in a solid body is $T(x, y, z)$, then its gradient vector field records the direction and rate of greatest increase in the temperature.
The *heat flow* is the vector field $\mathbf{F} = -k\nabla T$, where k (the *conductivity*) is a constant depending on the material.
The *heat flux* (= power output) out of a solid with boundary surface S is the flux of the heat flow:

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -k \iint_S \nabla T \cdot \mathbf{n} \, \mathrm{d}S$$

(Units: $[T]$ =K, $[\nabla T]$ =Km^{-1}, $[k]$ = Wm^{-1}K^{-1}, $[\mathbf{F}]$ =Wm^{-2})

Example A hot sphere of gold ($k = 318$ Wm^{-1}K^{-1}) of radius 10 cm has temperature

$$T(r) = \frac{1000}{1 + 10r} \quad \mathrm{K}$$

at a distance of $r = |\mathbf{r}|$ m from its center. The heat flow is then[5]

$$\mathbf{F} = \frac{3180000}{r(1 + 10r)^2}\mathbf{r} \quad \mathrm{Wm}^{-2}$$

On the surface of the sphere ($r = 0.1$ m) this is $\mathbf{F} = 7950000\mathbf{r}$ Wm^{-2}. Since the unit normal vector field is $10\mathbf{r}$, we find that the total heat flux of the gold sphere is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iint\limits_{r=0.1} 7950000\mathbf{r} \cdot 10\mathbf{r} \, \mathrm{d}S = 795000 \iint_{r=0.1} \mathrm{d}S = 795000 \cdot \frac{4\pi}{10^2} = 31800\pi \text{ W}$$

or approximately 100 kW.

[5]Using $\nabla f(r) = \frac{f'(r)}{r}\mathbf{r}$ for any radial function f.