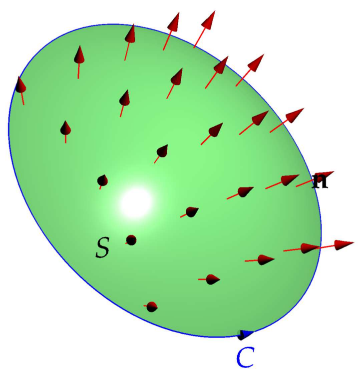
15.8 Stokes' Theorem

Recall how Green's Theorem could be written

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, \mathrm{d}A \tag{$*$}$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + 0\,\mathbf{k}$ and C is a simple closed curve in the plane $z = 0$ with interior D. Stokes' Theorem generalizes this to a curve which forms the boundary a surface in three dimensions.

To properly discuss this, we need to orient two types of objects:



Curves Orientation means the **direction of travel** along C.

Surfaces Orientation means the **direction of the normal vector field n**.

As pictured, both orientations can be made compatible in accordance with the right hand rule: curl the fingers of your right hand round C and your thumb will point (roughly) in the **n**-direction. We can also refer to the orientation *induced* on the (say) the boundary if this is compatible with the orientation of the surface.

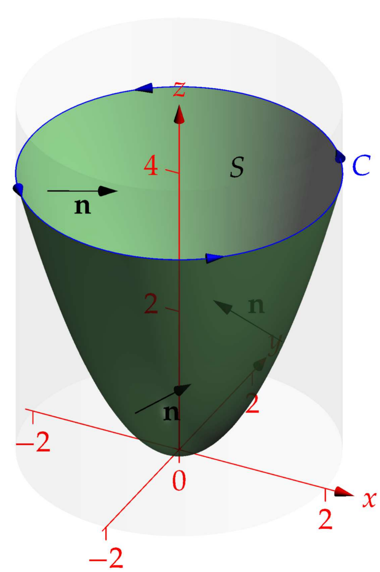
Theorem (Stokes)**.** *Let S be an oriented piecewise smooth surface with unit normal field* **n** *and piecewise smooth boundary curve $C = \partial S$ endowed with the induced orientation. Let* **F** *be a vector field with continuous partial derivatives on some open region containing S. Then*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

In two dimensions, a region D can be thought of a surface oriented upwards ($\mathrm{d}\mathbf{S} = \mathbf{k}\,\mathrm{d}A$); the induced orientation on its boundary C is counter-clockwise, recovering Green's Theorem ($*$).

Example 1 Compute the flux integral $\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where S is the part of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$ oriented upward, and $\mathbf{F}(x, y, z) = x^2z^2\,\mathbf{i} + y^2z^2\,\mathbf{j} + xyz\,\mathbf{k}$.

Evaluating the flux integral directly is nasty, so we instead apply Stokes' Theorem.



The boundary curve of S is the circle C of radius 2 in the plane $z = 4$. Parametrize this by

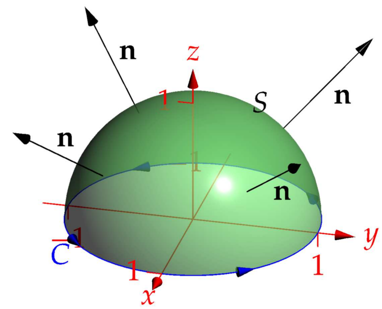
$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4\,\mathbf{k}, \quad 0 \le t \le 2\pi$$

Note that the counter-clockwise orientation of C is induced from the upward orientation of S. Now compute:

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \begin{pmatrix} 64\cos^2 t \\ 64\sin^2 t \\ 16\sin t\cos t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} \mathrm{d}t \\ &= \int_0^{2\pi} 128(\cos t\sin^2 t - \sin t\cos^2 t)\,\mathrm{d}t \\ &= \frac{128}{3}(\sin^3 t + \cos^3 t)\Big|_0^{2\pi} = 0\end{aligned}$$

Example 2 This time we check Stokes' Theorem directly. Let S be the unit hemisphere with $z \geq 0$ and $\mathbf{F} = 2x\mathbf{i} + (x - y^2)\mathbf{j} + xz^2\mathbf{k}$.

The boundary curve is the unit circle oriented counter-clockwise when viewed from above:



$$\mathbf{r}_C(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \leq t \leq 2\pi$$

We therefore have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \begin{pmatrix} 2\cos t \\ \cos t - \sin^2 t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt \\ &= \int_0^{2\pi} -2\sin t\cos t + \cos^2 t - \cos t \sin^2 t \, dt = \pi\end{aligned}$$

We can parametrize the surface in several ways to compute the flux integral, though it is helpful here first to observe that $\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ on this surface, so

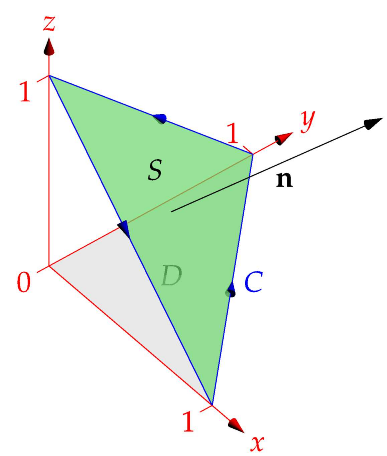
$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S \begin{pmatrix} 0 \\ -z^2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} dS = \iint_S z - yz^2 \, dS = \iint_S z \, dS$$

We used the fact that yz^2 is an *odd function* on the surface. At this point we can parametrize however we like. Using spherical polar co-ordinates, $dS = \sin\phi \, d\phi \, d\theta$ and

$$\iint_S z \, dS = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi \, d\phi \, d\theta = 2\pi \cdot \frac{1}{2} \sin^2 \phi \Big|_0^{\frac{\pi}{2}} = \pi$$

Example 3: Choosing the Surface If Stokes' Theorem is used to help evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ round a closed curve, remember that we can choose *any* surface S whose boundary is C!

If C is the triangle with corners $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ traced counter-clockwise when viewed from above, it seems sensible to *choose* S to be the *plane* $x + y + z = 1$ bounded by C.
Given $\mathbf{F} = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$, then



$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S \begin{pmatrix} -2z \\ -2x \\ -2y \end{pmatrix} \cdot d\mathbf{S} \\ &= \iint_D \begin{pmatrix} -2(1-x-y) \\ -2x \\ -2y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dx\,dy \\ &= -2 \iint_D dx\,dy = -1\end{aligned}$$

where we parametrized S using standard rectangular co-ordinates, and D is the base triangle $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$.

We could have chosen another surface S with boundary C: for instance $S = S_1 \cup S_2 \cup S_3$ as the union of three triangles in the co-ordinate planes $x = 0$, $y = 0$, $z = 0$, all oriented positively. You should check in this case that the resulting flux integral is unchanged:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{x=0} -2z \, dy\,dz + \iint_{y=0} -2x \, dx\,dz + \iint_{z=0} -2y \, dx\,dy = -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} = -1$$

Proving Stokes' Theorem

The conceptual part of the proof is little more than an application of Green's Theorem. The nasty part is checking all the algebra…

Sketch Proof. Let $\mathbf{r}(u,v)$ parametrize S with oriented co-ordinates, where $(u,v)\in D$. Let ∂D be the boundary curve of D oriented counter-clockwise. Then $\mathbf{r}(\partial D)=C$, and the orientations of these curves is compatible. Now compute:

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_{\partial D}\mathbf{F}\cdot(\mathbf{r}_u\,\mathrm{d}u+\mathbf{r}_v\,\mathrm{d}v)=\int_{\partial D}(\mathbf{F}\cdot\mathbf{r}_u)\,\mathrm{d}u+(\mathbf{F}\cdot\mathbf{r}_v)\,\mathrm{d}v\\
&=\iint_D \frac{\partial}{\partial u}(\mathbf{F}\cdot\mathbf{r}_v)-\frac{\partial}{\partial v}(\mathbf{F}\cdot\mathbf{r}_u)\,\mathrm{d}u\,\mathrm{d}v && \text{(Green's Theorem on } D\text{)}\\
&=\iint_D \mathbf{F}_u\cdot\mathbf{r}_v-\mathbf{F}_v\cdot\mathbf{r}_u\,\mathrm{d}u\,\mathrm{d}v && \text{(since } \mathbf{r}_{vu}=\mathbf{r}_{uv}\text{)}\\
&=\iint_D \begin{pmatrix}P_u\\Q_u\\R_u\end{pmatrix}\cdot\begin{pmatrix}x_v\\y_v\\z_v\end{pmatrix}-\begin{pmatrix}P_v\\Q_v\\R_v\end{pmatrix}\cdot\begin{pmatrix}x_u\\y_u\\z_u\end{pmatrix}\mathrm{d}u\,\mathrm{d}v && (*)\end{aligned}$$

Now compute from the other side:

$$\begin{aligned}\iint_S \nabla\times\mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iint_D(\nabla\times\mathbf{F})\cdot\big(\mathbf{r}_u\times\mathbf{r}_v\big)\,\mathrm{d}u\,\mathrm{d}v=\iint_D\left[\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\times\begin{pmatrix}P\\Q\\R\end{pmatrix}\right]\cdot\left[\begin{pmatrix}x_u\\y_u\\z_u\end{pmatrix}\times\begin{pmatrix}x_v\\y_v\\z_v\end{pmatrix}\right]\mathrm{d}u\,\mathrm{d}v\\
&=\iint_D\begin{pmatrix}R_y-Q_z\\P_z-R_x\\Q_x-P_y\end{pmatrix}\cdot\begin{pmatrix}y_uz_v-y_vz_u\\z_ux_v-z_vx_u\\x_uy_v-x_vy_u\end{pmatrix}\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

After dotting out and applying the chain rule ($P_u=P_xx_u+P_yy_u+P_zz_u$, etc.) to $(*)$, the two integrands are seen to be equal.

If S cannot be covered by a single parametrization, then a cut-and-paste argument similar to the proof of Green's Theorem can be used.

■

Stokes' Theorem and Conservative Vector Fields

Recall that every conservative vector field is irrotational:

$$\mathbf{F}=\nabla f \implies \nabla\times\mathbf{F}=\nabla\times\nabla f=\mathbf{0}$$

As previously advertised, Stokes' Theorem allows us to reverse this, provided the domain is simply-connected. As a loose justification, if C is any simple closed curve in the domain, let S be a surface whose boundary is C, then

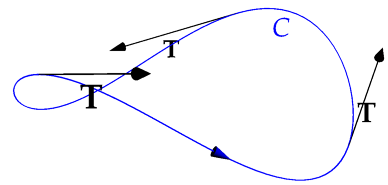
$$\int_C\mathbf{F}\cdot\mathrm{d}\mathbf{r}=\iint_S\nabla\times\mathbf{F}\cdot\mathrm{d}\mathbf{S}=0$$

To see why simple-connectedness is necessary, revisit the rotational example where $\mathbf{F}=\frac{1}{x^2+y^2}\begin{pmatrix}-y\\x\\0\end{pmatrix}$, whose domain is $\mathbb{R}^3$ with the z-axis removed. This has $\operatorname{curl}\mathbf{F}=\mathbf{0}$, but is non-conservative. Stokes' Theorem does not apply here since any surface with boundary C is necessarily cut by the z-axis.

Interpreting Curl: Circulation and Local Rotation

If C is a simple closed curve, the line integral

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s$$

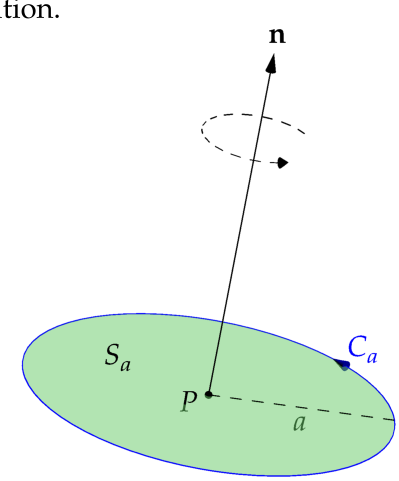


is also known as the *circulation* of $\mathbb{F}$ around C, since we are essentially computing the net component of $\mathbf{F}$ pointing in the direction of travel round the curve (in the direction of the unit tangent vector $\mathbf{T}$).

Stokes' Theorem lets us interpret the circulation round C in terms of the curl of $\mathbf{F}$: if S is *any* (suitable) surface with boundary C, then the circulation equals the flux integral of the curl,

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot \mathrm{d}\mathbf{S}$$

Otherwise said, the circulation of $\mathbf{F}$ around the boundary is the net effect of all the infinitesimal rotations induced by $\mathbf{F}$ on the surface. But what are these "infinitesimal rotations?" We mentioned this when first discussing curl, but we can now provide some justification.



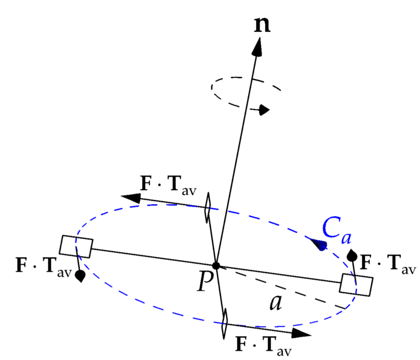
Let S_a be the disk of radius a centered at P with unit normal vector $\mathbf{n}$, and $\mathbf{F}$ a vector field satisfying Stokes' Theorem. Considering average values,

$$\int_{C_a} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_{S_a} (\nabla\times\mathbf{F})\cdot\mathbf{n}\,\mathrm{d}S = \pi a^2 (\nabla\times\mathbf{F})\cdot\mathbf{n}_{\mathrm{av}}$$

and taking the limit, we may interpret curl as the limit of the circulation per unit area:

$$(\nabla\times\mathbf{F})_P\cdot\mathbf{n} = \lim_{a\to 0^+}\frac{1}{\pi a^2}\int_{C_a}\mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s$$

Now consider a a small paddle of radius a and axis $\mathbf{n}$ rotating with the flow at P. The approximate angular speed of the paddle ω therefore satisfies



$$\begin{aligned} a\omega &= (\mathbf{F}\cdot\mathbf{T})_{\mathrm{av}} = \frac{1}{2\pi a}\int_{C_a}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \frac{\pi a^2}{2\pi a}(\nabla\times\mathbf{F})\cdot\mathbf{n}_{\mathrm{av}} \\ &= \frac{a}{2}(\nabla\times\mathbf{F})\cdot\mathbf{n}_{\mathrm{av}} \end{aligned}$$

Dividing by a and taking the limits as $a\to 0^+$, we see that an infinitesimal paddle at P will rotate with angular speed

$$\omega = \frac{1}{2}(\nabla\times\mathbf{F})_P\cdot\mathbf{n}\ \mathrm{rad/s}$$

The paddle therefore rotates with maximum velocity if its axis of rotation is parallel to the curl.