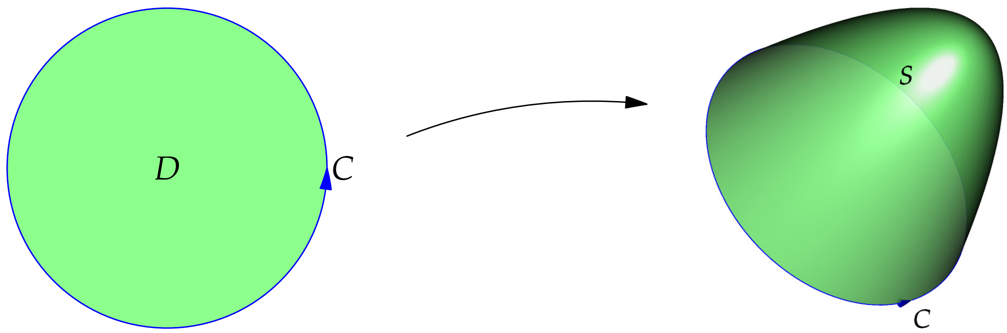
15.8 Stokes' Theorem

Stokes' theorem[1] is a three-dimensional version of Green's theorem. Recall the formula

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \,\mathrm{d}A$$

when $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$ and C is a simple closed curve in the plane $z = 0$ with interior D

Stokes' theorem generalizes this to curves which are the boundary of some part of a surface in three dimensions



Induced Orientation Orientation has meaning for curves *and* surfaces:

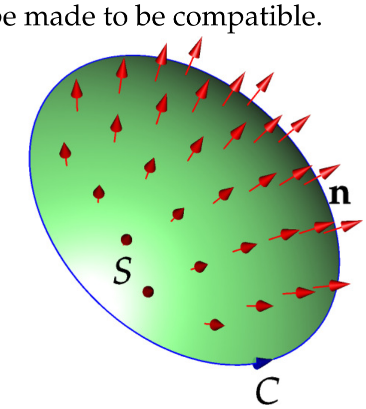
Curves Orientation = direction of travel along C.

Surfaces Orientation = direction of normal vector field $\mathbf{n}$.

If a curve is the boundary of a surface then the orientations of both can be made to be compatible.

Definition. *Let $C = \partial S$ be the boundary curve of an oriented surface S. C has the* induced orientation *if S is always to the left when traveling round C.*
Also say that S has the induced orientation *from C.*

To make sense of 'left' you must imagine that you are an inhabitant of the curve and that your head points in the $\mathbf{n}$ direction.



The Right Hand Rule Orientation works according to the Right Hand Rule:

Coil the fingers of your right hand and stick your thumb out
C follows your index finger while $\mathbf{n}$ points along your thumb.

[1] After George Stokes, hence Stokes' not Stoke's.

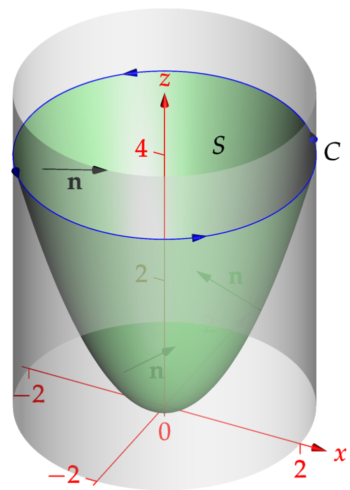
With orientation understood we can state:

Theorem (Stokes' Theorem)**.** *Let S be an oriented piecewise smooth surface with unit normal field $\mathbf{n}$ and piecewise smooth boundary curve $C = \partial S$ endowed with the induced orientation. Let $\mathbf{F}$ be a vector field with continuous partial derivatives on some open region containing S. Then*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Stokes' theorem relates a flux integral over a non-complete surface to a line integral around its boundary.

Example Compute the flux integral $\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where S is the part of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$ oriented upward, and $\mathbf{F}(x, y, z) = x^2z^2\mathbf{i} + y^2z^2\mathbf{j} + xyz\mathbf{k}$.



Rather than evaluating $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, we simply compute a line integral.

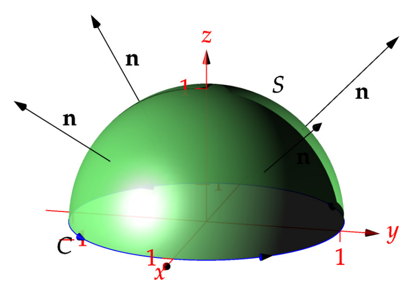
The boundary curve of S is the circle of radius 2 in the plane $z = 4$, parameterized by

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4\mathbf{k}, \quad 0 \le t \le 2\pi$$

Note that the counter-clockwise orientation of C is induced from the orientation of S. Now we compute:

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \begin{pmatrix} 64\cos^2 t \\ 64\sin^2 t \\ 16\sin t\cos t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} \mathrm{d}t \\ &= \int_0^{2\pi} 128(\cos t\sin^2 t - \sin t\cos^2 t)\,\mathrm{d}t \\ &= \frac{128}{3}(\sin^3 t + \cos^3 t)\Big|_0^{2\pi} = 0\end{aligned}$$

Example This time we check Stokes' Theorem directly. Let S be the unit hemisphere with $z \ge 0$ and $\mathbf{F} = 2x\mathbf{i} + (z^2 - x)\mathbf{j} + xz^2\mathbf{k}$. Clearly



$$\nabla \times \mathbf{F} = \begin{pmatrix} -2z \\ -z^2 \\ -1 \end{pmatrix}$$

Parameterizing the surface using spherical polar co-ordinates $(u, v) = (\phi, \theta)$, we obtain

$$\mathbf{r}(u, v) = \begin{pmatrix} \sin u\cos v \\ \sin u\sin v \\ \cos u \end{pmatrix} \implies \mathrm{d}\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v\,\mathrm{d}u\,\mathrm{d}v = \sin u \begin{pmatrix} \sin u\cos v \\ \sin u\sin v \\ \cos u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v$$

We therefore have

$$\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/2} \begin{pmatrix} -2\cos u \\ -\cos^2 u \\ -1 \end{pmatrix} \cdot \sin u \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^{2\pi} \int_0^{\pi/2} -2\cos u \sin^2 u \cos v - \cos^2 u \sin^2 u \sin v - \sin u \cos u \,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^{\pi/2} -2\pi \sin u \cos u \,\mathrm{d}u = \int_0^{\pi/2} -\pi \sin 2u \,\mathrm{d}u = \frac{\pi}{2}\cos 2u \Big|_0^{\pi/2} = -\pi$$

Alternatively, we could compute the line integral around C. This is easily done by parameterizing, but we can also appeal to the area-corollary of Green's Theorem and the the Fundamental Theorem of Line Integrals to avoid parameterizing: since $z = 0$ on C, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \begin{pmatrix} 2x \\ -x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ 0 \end{pmatrix} = \int_C 2x\,\mathrm{d}x - x\,\mathrm{d}y = \int_C \nabla x^2 \cdot \mathrm{d}\mathbf{r} - \mathrm{Area}(C) = -\pi$$

The proof of Stokes' Theorem is nothing more than an application of Green's Theorem in the parameterization space D.

Proof of Stokes' Theorem. Let $(u, v) \in D$ be oriented co-ordinates on S (parameterized by $\mathbf{r}(u, v)$). Now apply Green's Theorem to the region D:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot (\mathbf{r}_u\,\mathrm{d}u + \mathbf{r}_v\,\mathrm{d}v) = \int_{\partial D} (\mathbf{F} \cdot \mathbf{r}_u)\,\mathrm{d}u + (\mathbf{F} \cdot \mathbf{r}_v)\,\mathrm{d}v$$

$$= \iint_D \frac{\partial}{\partial u}(\mathbf{F} \cdot \mathbf{r}_v) - \frac{\partial}{\partial v}(\mathbf{F} \cdot \mathbf{r}_u)\,\mathrm{d}u\,\mathrm{d}v \qquad \text{(by Green's Theorem)}$$

$$= \iint_D \mathbf{F}_u \cdot \mathbf{r}_v - \mathbf{F}_v \cdot \mathbf{r}_u\,\mathrm{d}u\,\mathrm{d}v \qquad \text{(since } \mathbf{r}_{vu} = \mathbf{r}_{uv}\text{)}$$

$$= \iint_D \begin{pmatrix} P_u \\ Q_u \\ R_u \end{pmatrix} \cdot \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} - \begin{pmatrix} P_v \\ Q_v \\ R_v \end{pmatrix} \cdot \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v$$

Now apply the chain rule $\left(\frac{\partial P}{\partial u} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}, \text{ etc.}\right)$ and cancel/factor terms to obtain

$$= \iint_D \begin{pmatrix} P_x x_u + P_y y_u + P_z z_u \\ Q_x x_u + Q_y y_u + Q_z z_u \\ R_x x_u + R_y y_u + R_z z_u \end{pmatrix} \cdot \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} - \begin{pmatrix} P_x x_v + P_y y_v + P_z z_v \\ Q_x x_v + Q_y y_v + Q_z z_v \\ R_x x_v + R_y y_v + R_z z_v \end{pmatrix} \cdot \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \mathrm{d}u\,\mathrm{d}v$$

$$= \iint_D (R_y - Q_z)(y_u z_v - y_v z_u) + (P_z - R_x)(z_u x_v - z_v x_u) + (Q_x - P_y)(x_u y_v - x_v y_u)\,\mathrm{d}u\,\mathrm{d}v$$

$$= \iint_D \left[\nabla \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix}\right] \cdot \left[\begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \times \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}\right] \mathrm{d}u\,\mathrm{d}v$$

$$= \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{r}_u \times \mathbf{r}_v\,\mathrm{d}u\,\mathrm{d}v = \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

If S cannot be covered by a single parameterization, then a cut-and-paste argument similar to the proof of Green's Theorem can be used.

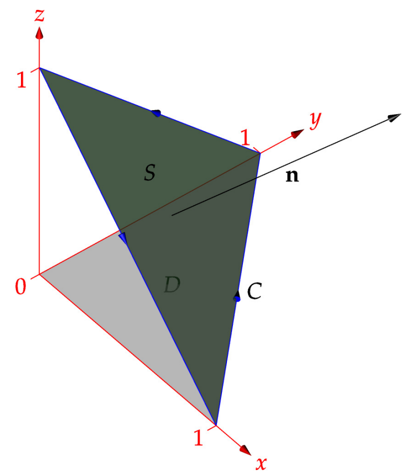
∎

Choosing the Surface S Since the line integral round a curve depends only on that curve and not on what surface the curve might be bounding, we have the following:

Corollary. $\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ *depends* only on the boundary ∂S *of* S *and not on its shape.*

This can be used to make a sensible choice of surface when trying to compute a line integral.

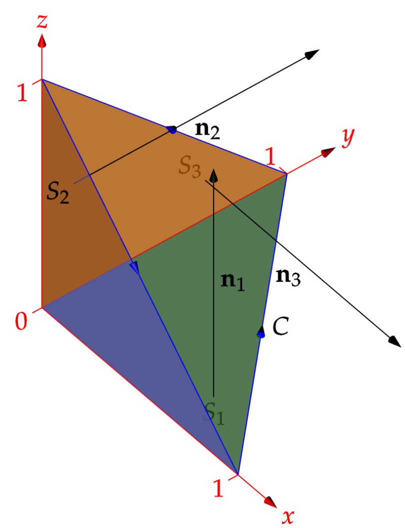
Example Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the triangle with corners $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ traced counter-clockwise from above, and $\mathbf{F} = (x+y^2)\mathbf{i} + (y+z^2)\mathbf{j} + (z+x^2)\mathbf{k}$



By the Corollary, we may apply Stokes' Theorem to *any* surface with boundary C. The obvious choice is the plane $z = 1-x-y$. Parameterizing as a graph $(u,v) = (x,y)$ we can see the region D as the base triangle of a tetrahedron. Whence

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \begin{pmatrix} -2z \\ -2x \\ -2y \end{pmatrix} \cdot \mathrm{d}\mathbf{S} \\ &= \iint_D \begin{pmatrix} -2(1-u-v) \\ -2u \\ -2v \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{d}u\,\mathrm{d}v \\ &= \iint_D -2(1-u-v) - 2u - 2v \,\mathrm{d}u\,\mathrm{d}v = -2\iint_D \mathrm{d}u\,\mathrm{d}v = -1\end{aligned}$$

We could have chosen S to be a different surface, such as the union $S_1 \cup S_2 \cup S_3$ of three parts of the co-ordinate planes as shown. While more work, each of the individual surface integrals is much simpler: each evaluates to $-\frac{1}{3}$ to give the same result as before:



$$\begin{aligned}\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}_1 &= \iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{k} \,\mathrm{d}x\,\mathrm{d}y = \iint_{S_1} -2y \,\mathrm{d}x\,\mathrm{d}y \\ \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}_2 &= \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{j} \,\mathrm{d}x\,\mathrm{d}z = \iint_{S_2} -2x \,\mathrm{d}x\,\mathrm{d}z \\ \iint_{S_3} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}_3 &= \iint_{S_3} \nabla \times \mathbf{F} \cdot \mathbf{i} \,\mathrm{d}y\,\mathrm{d}z = \iint_{S_3} -2z \,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

$$\implies \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 3\left(\frac{-1}{3}\right) = -1$$

Stokes' Theorem for Conservative Fields Let $\mathbf{F} = \nabla f$ be conservative: we know that conservative fields are irrotational. Indeed

$$\nabla \times \mathbf{F} = \nabla \times \nabla f = 0 \implies \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0 \quad \text{for } any \text{ surface } S$$

Alternatively, the boundary ∂S of any surface S is a closed curve, and so, by the conservatism of $\mathbf{F}$,

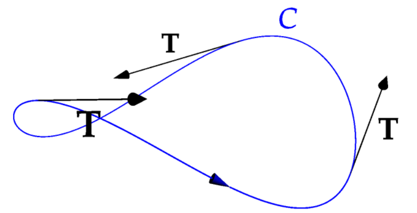
$$\int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

Stokes' Theorem simply says that $0 = 0$.

Interpretation of Curl: Circulation

When a vector field $\mathbf{F}$ is a velocity field,[2] Stokes' Theorem can help us understand what curl means.

Recall: If t is any parameter and s is the arc-length parameter then



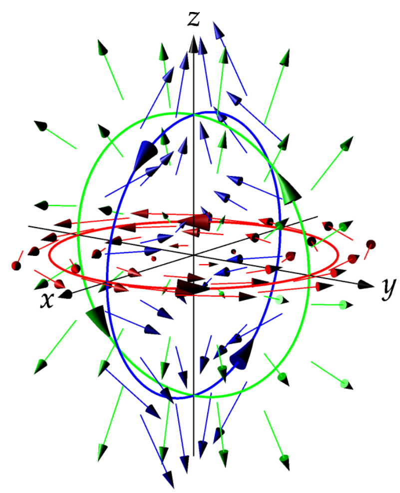
$$d\mathbf{r} = \frac{d\mathbf{r}}{dt}\,dt = \frac{d\mathbf{r}/dt}{ds/dt}\,ds = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,ds = \mathbf{T}\,ds$$

where $\mathbf{T}$ is the unit tangent vector of C.

Definition. *Let C be a closed curve and $\mathbf{v}$ a vector field. The integral $\int_C \mathbf{v}\cdot d\mathbf{r} = \int_C \mathbf{v}\cdot\mathbf{T}\,ds$ is the* circulation *of $\mathbf{v}$ around C.*

Stokes' Theorem $\implies$ Circulation $= \iint_S \nabla\times\mathbf{v}\cdot d\mathbf{S}$ where S is *any* surface with boundary C.

Example With a plot, we can visualize and calculate circulation. We calculate the circulation of $\mathbf{v} = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$ around circles of radius a centered at the origin in each of the three co-ordinate planes, where each circle has orientation inherited from $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively



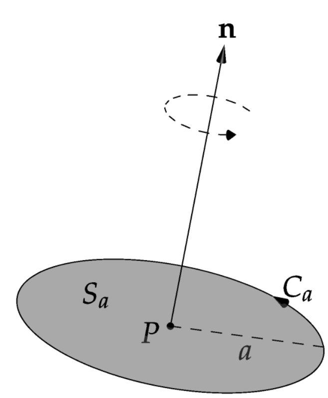
First notice that $\nabla\times\mathbf{v} = 2\mathbf{k}$. The circulations can then be computed:

$$\left.\begin{array}{l} \nabla\times\mathbf{v}\cdot\mathbf{i}\,dS|_{x=0} = 0 \\ \nabla\times\mathbf{v}\cdot\mathbf{j}\,dS|_{y=0} = 0 \\ \nabla\times\mathbf{v}\cdot\mathbf{k}\,dS|_{z=0} = 2\,dx\,dy \end{array}\right\} \implies \int_C \mathbf{v}\cdot d\mathbf{r} = \left\{\begin{array}{l} 0 \\ 0 \\ 2\pi a^2 \end{array}\right.$$

It should be obvious from the plot why the circulations are zero, zero, positive, in that order.

Abstract consideration Now we apply Stokes' Theorem to the concept of curl in an abstract sense.

Let S_a be the disk of radius a centered at P and with unit normal vector $\mathbf{n}$, and $\mathbf{v}$ a vector field satisfying Stokes' Theorem.



The circulation of $\mathbf{v}$ around the oriented boundary C_a is

$$\int_{C_a} \mathbf{v}\cdot d\mathbf{r} = \iint_{S_a} (\nabla\times\mathbf{v})\cdot\mathbf{n}\,dS = \pi a^2 (\nabla\times\mathbf{v})\cdot\mathbf{n}_{\text{av}}$$

where $(\nabla\times\mathbf{v})\cdot\mathbf{n}_{\text{av}}$ is the average value over S_a, and πa^2 is the area of S_a. Hence

$$(\nabla\times\mathbf{v})\cdot\mathbf{n}_{\text{av}} = \frac{1}{\pi a^2}\int_{C_a} \mathbf{v}\cdot d\mathbf{r}$$

This can be thought of as the circulation per unit area.

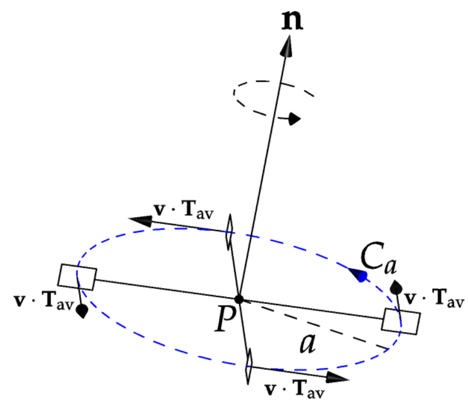
[2] $\mathbf{F}(P)$ is the velocity of a particle at point P.

Since $\mathbf{v}$ has continuous partial derivatives, we may take the limit as $a \to 0$ to obtain

$$(\nabla \times \mathbf{v})(P) \cdot \mathbf{n} = \lim_{a\to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot \, \mathrm{d}\mathbf{r} = \lim_{a\to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot \mathbf{T} \, \mathrm{d}s$$

The component of the curl at P in the direction $\mathbf{n}$ is therefore the limit of the circulation per unit area around $\mathbf{n}$.

We can apply a similar analysis to the circulation integral. Imagine a small paddle of radius a and axis $\mathbf{n}$ rotating with the flow at P.



$(\mathbf{v} \cdot \mathbf{T})_{\text{av}}$ is the approximate linear speed of the paddle at radius a, whence the approximate angular speed of the paddle is

$$\omega = \frac{1}{a}(\mathbf{v} \cdot \mathbf{T})_{\text{av}}$$

with improving approximation as $a \to 0$. Since a line integral is merely the length of the curve multiplied by the average value of the integrand, we have

$$\begin{aligned}(\nabla \times \mathbf{v})(P) \cdot \mathbf{n} &= \lim_{a\to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot \mathbf{T} \, \mathrm{d}s = \lim_{a\to 0} \frac{1}{\pi a^2} \cdot 2\pi a (\mathbf{v} \cdot \mathbf{T})_{\text{av}} \\ &= \lim_{a\to 0} \frac{2\pi a \cdot a\omega}{\pi a^2} = 2\omega\end{aligned}$$

The curl at P in direction $\mathbf{n}$ therefore measures *twice* the angular velocity of the paddle.

It follows that a paddle placed at P will rotate with maximum velocity if its axis of rotation is parallel to the curl.

Example A tiny paddle is placed at the point $(0, 1, 0)$ with vertical axis $\mathbf{k}$ in a river with flow $\mathbf{F} = (1 - \frac{1}{4}y^2)\mathbf{i}$. Its angular speed will be

$$\frac{1}{2} \nabla \times \mathbf{F} \cdot \mathbf{k} = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}y \end{pmatrix} \cdot \mathbf{k} = \frac{1}{4} \text{ rad/s.}$$

This is how the duck race examples from the curl section were calculated.