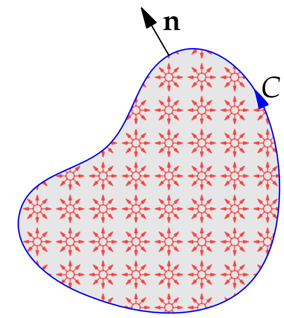
## 16.9 The Divergence Theorem

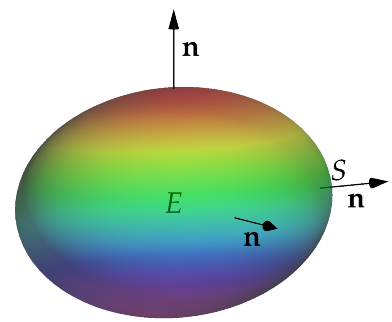
Recall that the flux of a vector field across a positively-oriented simple closed curve $C$ equals the integral of the divergence over its interior

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \iint_D \nabla\cdot\mathbf{F}\,\mathrm{d}A$$




The Divergence Theorem is nothing more than the same result for surfaces bounding volumes.

The Theorem applies to specific types of volume $E$ which can be visualized as distorted spheres.




1. $E$ must be bounded: in fits inside some box $\{\mathbf{r} : |\mathbf{r}| \le k\}$ where $k$ is a positive constant.
2. $E$ has a complete, closed boundary surface $S$: one cannot get into or out of $E$ without crossing the boundary $S$.
3. The boundary $S$ is oriented *outwards* from $E$.

Solid spheres, ellipsoids, cuboids, tetrahedra, etc., are all suitable candidates.

**Theorem** (Divergence/Gauss'/Ostrogradsky's Theorem)**.** *Let $E$ be a bounded volume with outward-orented complete, closed boundary surface $S$. If $\mathbf{F}$ is a vector field with continuous partial derivatives then*

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\,\mathrm{d}V$$

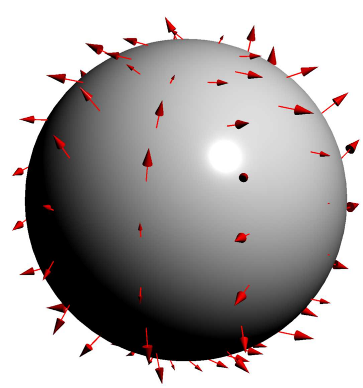
**Example 1** The $E$ be the ball of radius $R$ centered at the origin. Observe that $\mathbf{n} = \frac{1}{R}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$ on the boundary surface.

(a) For the radial vector field $\mathbf{F}_a(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, we easily verify

$$\iint_S \mathbf{F}_a\cdot\mathrm{d}\mathbf{S} = \iint_S R\,\mathrm{d}S = 4\pi R^3 = \iiint_E \nabla\cdot\mathbf{F}_a\,\mathrm{d}V = \iiint_E 3\,\mathrm{d}V$$

(b) Now consider the field $\mathbf{F}_b(x,y,z) = xz^2\,\mathbf{i} + x^2y\,\mathbf{j} + yz^2\,\mathbf{k}$. This time $\nabla\cdot\mathbf{F} = z^2 + x^2 + y^2 = r^2$, and

$$\iint_S \mathbf{F}_b\cdot\mathrm{d}\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}_b\,\mathrm{d}V = \iiint_E r^2\,\mathrm{d}V$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\cdot r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{4}{5}\pi R^5$$



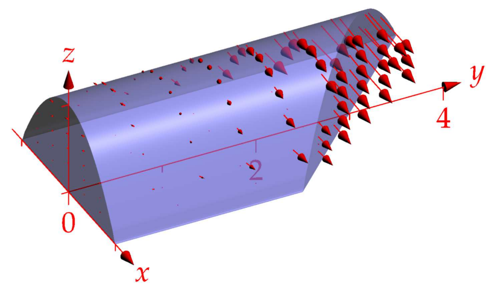
The alternative is to compute directly:

$$\iint_S \mathbf{F}_b\cdot\mathrm{d}\mathbf{S} = \iint_S \begin{pmatrix} xz^2 \\ yx^2 \\ zy^2 \end{pmatrix}\cdot\begin{pmatrix} x/R \\ y/R \\ z/R \end{pmatrix}\mathrm{d}S = \cdots$$

which is more time-consuming.

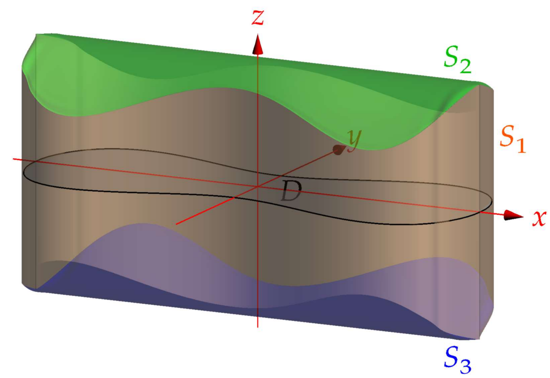
**Example 2** Triple integrals are often easier than flux integrals, particularly when a boundary surface is piecewise smooth and would therefore require multiple separate parametrizations.

Compute the net flux of the field $\mathbf{F} = y^2\,\mathbf{i} + (x^2 - z)\,\mathbf{j} + z^2\,\mathbf{k}$ out of the region $E$ bounded by the surface $z = 1 - x^2$ and the planes $y = 0$, $z = 0$ and $x + y = 3$.




$$\iint_{\partial E} \mathbf{F}\cdot d\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\,dV = \iiint_E 2z\,dV$$
$$= \int_{-1}^{1}\int_0^{3-x}\int_0^{1-x^2} 2z\,dz\,dy\,dx$$
$$= \int_{-1}^{1}\int_0^{3-x} (1-x^2)^2\,dy\,dx = \frac{16}{5}$$

**Proving the Divergence Theorem**




The proof structure is almost identical to that of Green's Theorem. We first prove for a special type of region, and then patch such together.

Suppose $E$ is a region lying between two graphs

$$(x, y) \in D, \quad g(x, y) \le z \le f(x, y)$$

Its boundary surface $S$ consists of three pieces:

1. The cylinder $S_1$ comprising all the points in $E$ for which $(x, y)$ lies on the boundary curve of $D$.
2. The upper surface $S_2$, being the graph of $z = f(x, y)$.
3. The lower surface $S_3$, the graph of $z = g(x, y)$.

Suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ satisfies the hypotheses. We compare the $R$-parts of the two integrals in the Theorem. Firstly,

$$\iiint_E \frac{\partial R}{\partial z}\,dV = \iint_D \left[\int_{g(x,y)}^{f(x,y)} \frac{\partial R}{\partial z}\,dz\right] dx\,dy = \iint_D R\big(x,y,f(x,y)\big) - R\big(x,y,g(x,y)\big)\,dx\,dy$$

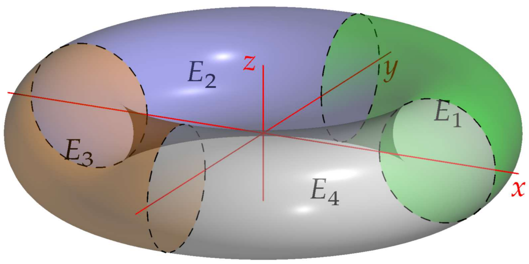
Now compute the flux integrals. Noting that $S_1$ is oriented *perpendicular* to $R\,\mathbf{k}$ ($R\mathbf{k}\cdot d\mathbf{S}_1 = 0$), and that $S_3$ is oriented *downwards*,

$$\iint_S \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot d\mathbf{S} = \iint_{S_1} \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot d\mathbf{S}_1 + \iint_{S_2} \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot d\mathbf{S}_2 + \iint_{S_3} \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot d\mathbf{S}_3$$
$$= \iint_{S_1} 0\,dS_1 + \iint_D \begin{pmatrix}0\\0\\R(x,y,f(x,y))\end{pmatrix}\cdot\begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix} dx\,dy + \iint_D \begin{pmatrix}0\\0\\R(x,y,g(x,y))\end{pmatrix}\cdot\begin{pmatrix}g_x\\g_y\\-1\end{pmatrix} dx\,dy$$
$$= \iint_D R\big(x,y,f(x,y)\big) - R\big(x,y,g(x,y)\big)\,dx\,dy$$

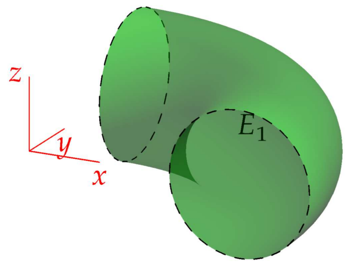
which is exactly the integral we saw before! If $E$ can also be described as lying between graphs $j(x,z) \le y \le h(x,z)$ and $l(y,z) \le x \le k(y,z)$, then the $P$- and $Q$-parts of the volume and surface integrals are also equal, and the Theorem is proved! This is true for simple examples such as cubes or spheres.

If $E$ cannot be described as lying between pairs of graphs in all three ways, we cut it into sub-volumes $E_1, \ldots, E_n$ each of which can be. Since interior faces are integrated twice and with opposite orientation, all interior surface integrals cancel and we get the Theorem.

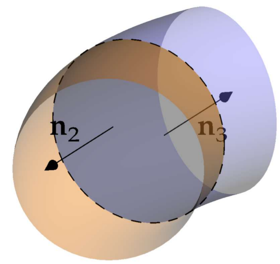
Rigorously showing that such a subdivision is always possible is beyond these notes. Instead, as an example, consider how the pictured torus has been cut into four sub-volumes $E_1, E_2, E_3, E_4$. The linked picture shows $E_1$: you should be able to convince yourself that it can be viewed as lying between graphs in all three co-ordinate directions.




Solid Torus $E$




Sub-volume of all three types




Flux integrals over interior faces cancel

To complete the argument, sum the triple integrals over each of the sub-volumes,

$$\iiint_E \nabla \cdot \mathbf{F} \, \mathrm{d}V = \left( \iiint_{E_1} + \cdots + \iiint_{E_4} \right) \nabla \cdot \mathbf{F} \, \mathrm{d}V = \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_1 + \cdots + \iint_{S_4} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_4$$

noting that the flux integrals over the shared (disk) boundaries are integrated twice with opposite orientations and thus cancel.

### Incompressible Fields

For an incompressible field ($\operatorname{div} \mathbf{F} = 0$) the flux across *any* complete bounded surface is

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, \mathrm{d}V = 0$$

Supposing that $S = S_1 \cup S_2$ is the boundary of $E$ and that $\mathbf{F}$ is incompressible, we obtain an analogue of Green's Theorem with holes:

$$\left( \iint_{S_1} + \iint_{S_2} \right) \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0 \implies \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -\iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Otherwise said, the flux *out* of $E$ across $S_1$ equals the flux *into* $E$ across $S_2$.

**Example 1** Compute the flux of the vector field $\mathbf{F} = y^2 \, \mathbf{i} - 3xz \, \mathbf{j} - 4 \, \mathbf{k}$ across the upward-oriented paraboloidal surface $z = 8 - x^2 - 2y^2$ where $z \geq 0$.

Parametrizing this surface and computing the flux integral directly would be nasty. However, $\nabla \cdot \mathbf{F} = 0$ so we can compare to the flux integral across the elliptic disk ($D$) $x^2 + 2y^2 = 8$ in the $xy$-plane.

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -\iint_D \begin{pmatrix} y^2 \\ -3xz \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S = -4 \iint_D \mathrm{d}A = 4\pi\sqrt{8}\sqrt{4} = 16\sqrt{2}\pi$$

**Example 2** Inverse square force laws are common in Physics: both gravitation and electromagnetism have the form $\mathbf{F} = \frac{k}{r^3}\mathbf{r}$ (magnitude $|\mathbf{F}| = \frac{k}{r^2}$). In three-dimensions, such fields are incompressible:[1]

$$\nabla \cdot \frac{k}{r^{-3}}\mathbf{r} = k\left(\nabla r^{-3} \cdot \mathbf{r} + r^{-3}\nabla \cdot \mathbf{r}\right) = k\left(\frac{-3r^{-4}}{r}\mathbf{r} \cdot \mathbf{r} + 3r^{-3}\right) = 0$$

The upshot is that the net flux out of any region *not containing the origin* is zero. By comparing to a small sphere $S_a$ centered at the origin (oriented outward), the flux across *any complete surface $S$ enclosing the origin* is seen to be identical!

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_a} \frac{k}{r^3}\mathbf{r} \cdot \frac{1}{r}\mathbf{r}\, dS = \iint_{S_a} \frac{k}{a^2}\, dS = 4\pi k \qquad (*)$$

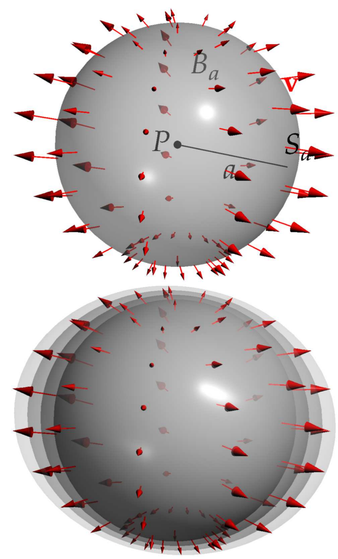
The converse is also true (though it needs a little knowledge from differential equations): the only radially symmetric fields with such properties are inverse-square fields.

### Interpreting Divergence (just for fun)

Similarly to our discussion of Stokes' Theorem, we repeat our "average" analysis for divergence.

Let $S_a$ be the sphere of radius $a$ centered at a point $P$, and denote the interior solid ball $B_a$. If the $\mathbf{F}$ satisfies the Divergence Theorem, then

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \nabla \cdot \mathbf{F}\, dV = V_a (\nabla \cdot \mathbf{F})_{av}$$




where $(\nabla \cdot \mathbf{F})_{av}$ is the average divergence of $\mathbf{F}$ over $B_a$ and $V_a = \frac{4}{3}\pi a^3$ is the volume of $B_a$. The divergence at a point is therefore the limit of the outward flux per unit volume:

$$(\nabla \cdot \mathbf{F})(P) = \lim_{a \to 0^+} \frac{1}{V_a} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

Now suppose the surface $S_a$ drifts with the field $\mathbf{F}$. If $t$ is small, the surface is still approximately spherical; it its average radius is $r(t)$, then $r'(0) = (\mathbf{F} \cdot \mathbf{n})_{av}$. Its rate of change of volume is therefore

$$\left.\frac{dV_a}{dt}\right|_{t=0} \approx \left.\frac{d}{dt}\right|_{t=0} \frac{4}{3}\pi r_{av}^3(t) = 4\pi r_{av}^2(0) r_{av}'(0) = 4\pi a^2 (\mathbf{F} \cdot \mathbf{n})_{av} = \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} \approx V_a\big((\nabla \cdot \mathbf{F})(P)\big)$$

Taking limits as $a \to 0$, we see that divergence is the (unitless) rate of change of infinitesimal volume:

$$\nabla \cdot \mathbf{F}(P) = \frac{1}{V}\frac{dV}{dt}$$

**Definition.** If $\nabla \cdot \mathbf{F}(P) > 0$, we call the point $P$ a *source*. A *sink* is where $\nabla \cdot \mathbf{F}(P) < 0$.

We can even interpret vector fields that are undefined this way, for example the inverse-square field $\mathbf{F} = r^{-3}\mathbf{r}$ can be viewed as having an infinite source at the origin:

$$(\nabla \cdot \mathbf{F})(O) := \lim_{a \to 0^+} \frac{1}{V_a} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \lim_{a \to 0^+} \frac{3}{4\pi a^3} \cdot 4\pi = +\infty$$

[1] Two facts are useful here: $\nabla f(r) = \frac{f'(r)}{r}\mathbf{r}$ and $\nabla \cdot f\mathbf{F} = (\nabla f) \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$.