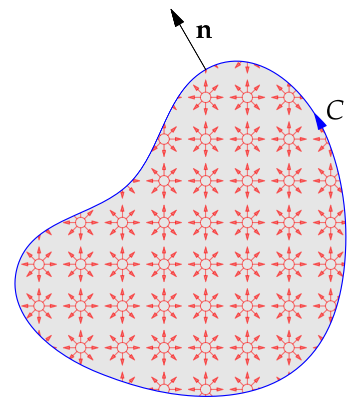
15.9 The Divergence Theorem

The Divergence Theorem is the second 3-dimensional analogue of Green's Theorem.

Recall: if **F** is a vector field with continuous derivatives defined on a region $D \subseteq \mathbb{R}^2$ with boundary curve C, then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \iint_D \nabla \cdot \mathbf{F} \, \mathrm{d}A$$

The flux of **F** across C is equal to the integral of the divergence over its interior.



The Divergence Theorem is nothing more than the same result for surfaces bounding volumes.

Notation/Orientation The Theorem applies to specific types of volumes E which can be imagined as distorted spheres.[1] Specifically:

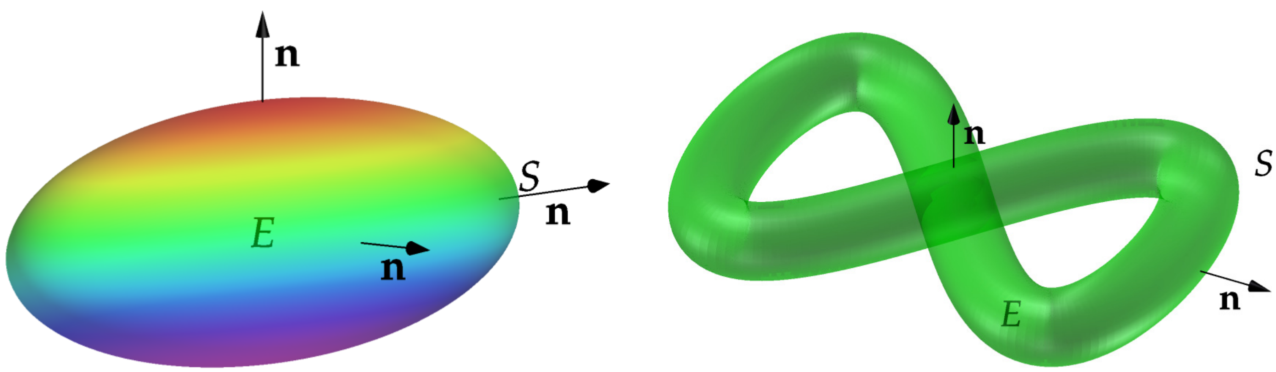
1. E must be bounded: E fits inside a box $\{\mathbf{r} : |\mathbf{r}| < k\}$ for some $k \in \mathbb{R}^+$.
2. E has a complete, closed boundary surface $S = \partial E$: one cannot get into or out of E without crossing the boundary S.
3. The boundary S is oriented *outwards* from E.

Spheres, Ellipsoids, Cuboids, Tetrahedra, etc. are all suitable candidates.

Unsuitable regions include:

The positive octant: $E = \{(x, y, z) : x, y, z \geq 0\}$ is unbounded.

Rational points in a cube: $E = \{(x, y, z) \in \mathbb{Q}^3 : 0 \leq x, y, z \leq 1\}$ has no sensible boundary.



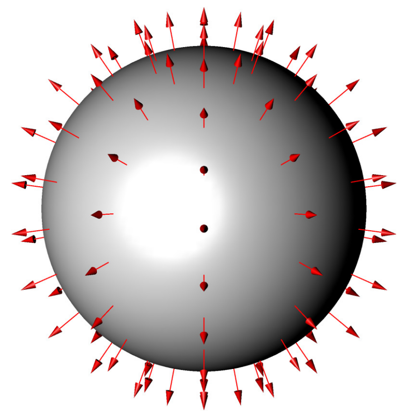
Suitable regions for the Divergence Theorem

[1] Or unions of disconnected such.

Theorem (Divergence/Gauss'/Ostrogradsky's Theorem). *Suppose that E is a bounded volume with complete, closed boundary surface S oriented outwards. If* **F** *is a vector field with continuous partial derivatives then*

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\,\mathrm{d}V$$

Example Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and E the ball of radius a centered at the origin



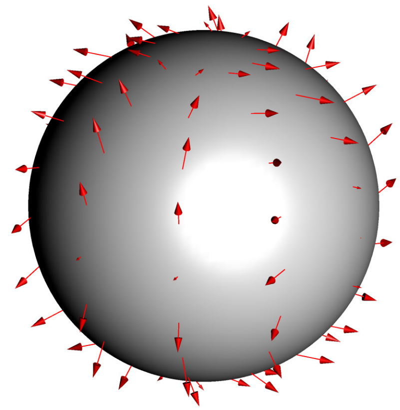
$\nabla\cdot\mathbf{F} = 3$, whence

$$\iiint_E \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint_E 3\,\mathrm{d}V = 4\pi a^3$$

Alternatively, S is the sphere with normal field $\mathbf{n} = \frac{1}{a}\begin{pmatrix} x\\ y\\ z\end{pmatrix}$, so we compare,

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S \begin{pmatrix} x\\ y\\ z\end{pmatrix}\cdot\begin{pmatrix} x/a\\ y/a\\ z/a\end{pmatrix}\,\mathrm{d}S = \iint_S \frac{x^2+y^2+z^2}{a}\,\mathrm{d}S = \iint_S a\,\mathrm{d}S = 4\pi a^3$$

Example Let $\mathbf{F}(x,y,z) = xz^2\mathbf{i} + yx^2\mathbf{j} + zy^2\mathbf{k}$ where E is again the ball of radius a centered at the origin.



Here $\nabla\cdot\mathbf{F} = z^2+x^2+y^2 = r^2$, and so

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iiint_E \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iiint_E r^2\,\mathrm{d}V$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\cdot r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{4}{5}\pi a^5$$

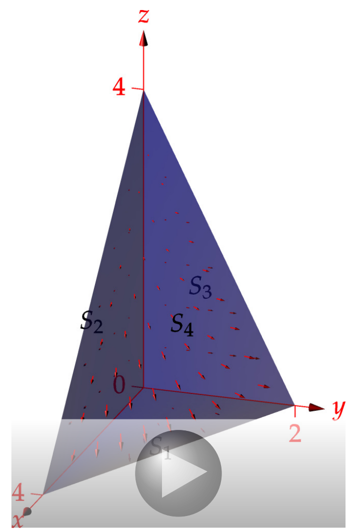
The alternative is to compute directly:

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S \begin{pmatrix} xz^2\\ yx^2\\ zy^2\end{pmatrix}\cdot\begin{pmatrix} x/a\\ y/a\\ z/a\end{pmatrix}\,\mathrm{d}S = \iint_S \frac{x^2z^2+y^2x^2+z^2y^2}{a}\,\mathrm{d}S = \cdots$$

which is very time consuming.

The Divergence Theorem often makes things much easier, in particular when a boundary surface is piecewise smooth. In the following example, the flux integral requires computation and parameterization of four different surfaces. Thanks to the Divergence Theorem the flux is merely a triple integral over a very simple region.

Example Let $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} - x^2y\mathbf{k}$ where E is the solid tetrahedron bounded by $x + 2y + z = 4$ and the co-ordinate planes. There are four surfaces, labeled in the picture. Parameterizing each of these would be time-consuming (*try it!*).

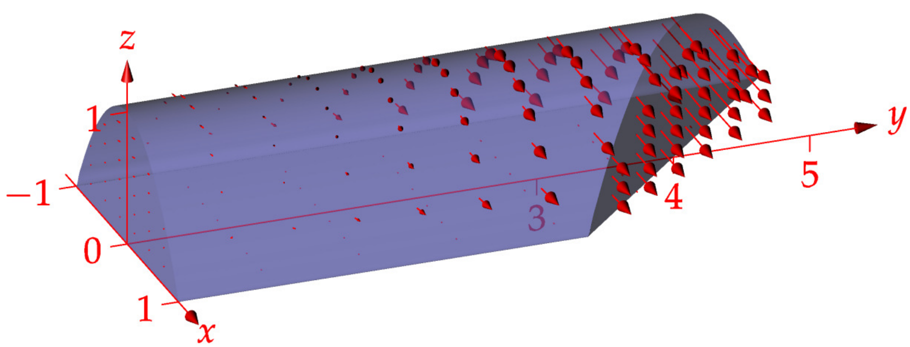


However, $\nabla \cdot \mathbf{F} = 3$ so, by the Divergence Theorem

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\, dV = 3\iiint_E dV$$
$$= 3 \cdot \frac{1}{3}\text{Area(base)} \cdot \text{Height} = \frac{1}{2} \cdot 4 \cdot 2 \cdot 4 = 16$$

Example E is bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, and $x + y = 4$
Find the flux of $\mathbf{F} = y^2\mathbf{i} + (x^2 - z)\mathbf{j} + z^2\mathbf{k}$ across ∂E

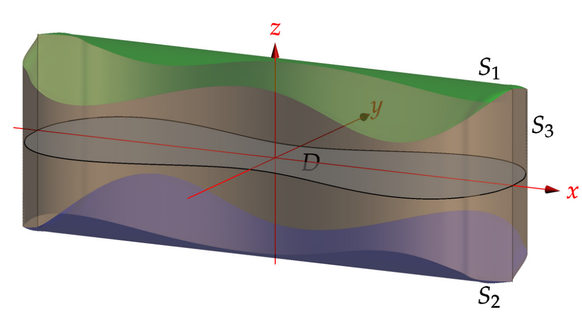
$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\, dV = \iiint_E 2z\, dV = \int_{-1}^{1}\int_0^{4-x}\int_0^{1-x^2} 2z\, dz\, dy\, dx$$
$$= \int_{-1}^{1}\int_0^{4-x} (1-x^2)^2\, dy\, dx = \int_{-1}^{1} (1-x^2)^2(4-x)\, dx = 8\int_0^1 (1-x^2)^2\, dx$$
$$= 8\int_0^1 1 - 2x^2 + x^4\, dx = 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}$$



Proving the Divergence Theorem

The proof is almost identical to that of Green's Theorem. We prove for different types of regions then perform a cut-and-paste argument.

The first type of region shown in the graphic. E lies between two graphs: $(x, y) \in D$ and $g(x, y) \le z \le f(x, y)$. Its boundary surface consists of three pieces: the graphs S_1 and S_2, and the cylindrical piece S_3.



Proof of Divergence Theorem. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and that E can be described as lying between two graphs: $g(x,y) \le z \le f(x,y)$ for $(x,y) \in D$. Let S_1 be the upper graph, S_2 the lower, and S_3 the sides.
We compare the R-parts of the two integrals in the Theorem. Firstly,

$$\iiint_E \frac{\partial R}{\partial z}\,\mathrm{d}V = \iint_D \left[\int_{g(x,y)}^{f(x,y)} \frac{\partial R}{\partial z}\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y = \iint_D R(x,y,f(x,y)) - R(x,y,g(x,y))\,\mathrm{d}x\,\mathrm{d}y$$

Now compute the flux integrals. Noting that S_2 is oriented *downward*, and S_3 at right angles to $R\mathbf{k}$ (hence $R\mathbf{k}\cdot\mathrm{d}\mathbf{S}_3 = 0$), we have,

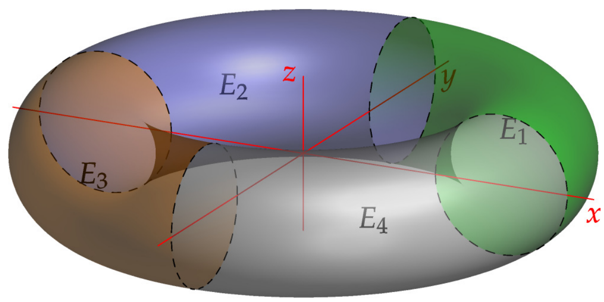
$$\begin{aligned}\iint_S \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot\mathrm{d}\mathbf{S} &= \iint_{S_1} \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot\mathrm{d}\mathbf{S}_1 + \iint_{S_2} \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot\mathrm{d}\mathbf{S}_2 + \iint_{S_3} \begin{pmatrix}0\\0\\R\end{pmatrix}\cdot\mathrm{d}\mathbf{S}_3\\ &= \iint_D \begin{pmatrix}0\\0\\R(x,y,f(x,y))\end{pmatrix}\cdot\begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix}\mathrm{d}x\,\mathrm{d}y + \iint_D \begin{pmatrix}0\\0\\R(x,y,g(x,y))\end{pmatrix}\cdot\begin{pmatrix}g_x\\g_y\\-1\end{pmatrix}\mathrm{d}x\,\mathrm{d}y\\ &= \iint_D R(x,y,f(x,y)) - R(x,y,g(x,y))\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

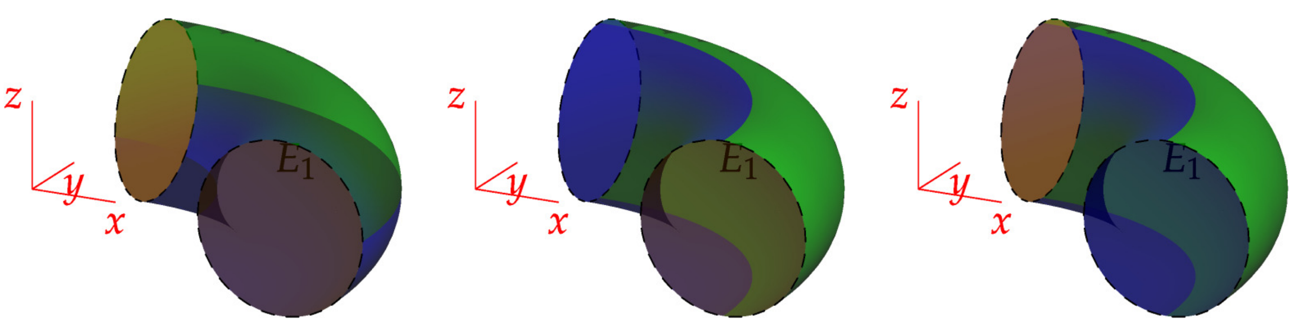
which is exactly the integral we saw before.

If we can also describe E as lying between graphs $j(x,z) \le y \le h(x,z)$ and $l(y,z) \le x \le k(y,z)$ then it quickly follows that the P- and Q-parts of the volume and surface integrals are equal, and that the Theorem is proved. This is certainly true for examples such as cubes or spheres.

If E cannot be described as lying between pairs of graphs in all three ways, we cut it into sub-volumes $E_1, \ldots, E_n$ each of which can be. Since interior faces are integrated twice and with opposite orientation, all interior surface integrals cancel and we get the Theorem. ■

It requires some thinking to convince yourself that any volume can be sub-divided in the required way. As an example, consider a torus, which we've cut into four regions E_1, E_2, E_3, E_4. First we convince ourselves that E_1 is of the required type. The other regions are similar.



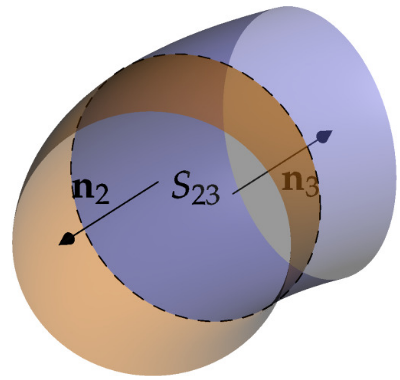


Upper graph: $z = f(x,y)$
Lower graph: $z = g(x,y)$
Two Sides

Right graph: $x = k(y,z)$
Left graph: $x = l(y,z)$
One Side

Front graph: $y = h(x,z)$
Rear graph: $y = j(x,z)$
One Side

Now label the boundary surface of each region E_n by S_n, and the boundary surface between E_m, E_n by S_{mn}. The graphic shows S_{23} for the torus. Note that $S_{23} = S_2 \cup S_3$ is oriented in the direction $\mathbf{n}_2$ when considered as part of S_2 and in the opposite direction $\mathbf{n}_3 = -\mathbf{n}_2$ when part of S_3. The divergence theorem certainly applies to each solid E_n, whence



$$\iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}V = \left(\iiint_{E_1} + \cdots + \iiint_{E_4} \right) \nabla \cdot \mathbf{F}\,\mathrm{d}V$$
$$= \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_1 + \cdots + \iint_{S_4} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_4$$

The flux integrals $\iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_2$ and $\iint_{S_3} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_3$ have the surface S_{23} in common, but each computes the flux in the opposite direction. The net contribution of S_{23} is therefore zero. The same reasoning applies to the other internal surface. It follows that the sum of the flux integrals is indeed the total flux out of E: $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, as required.

Incompressible Fields

For incompressible fields ($\operatorname{div} \mathbf{F} = 0$) we have

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}V = 0$$

over every complete bounded surface $S = \partial E$. I.e. incompressible fields have zero net flux out of all regions.

Suppose that $S = S_1 \cup S_2$ is the boundary of E, and that $\mathbf{F}$ is incompressible. Then

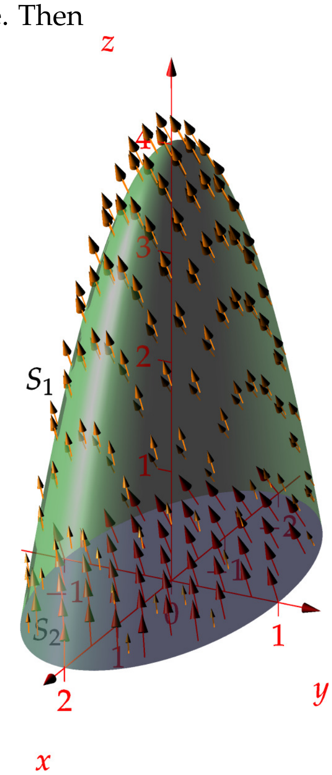
$$\left(\iint_{S_1} + \iint_{S_2} \right) \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0 \implies \iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -\iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Thus the flux *out* across S_1 equals the flux *in* across S_2.

For example, recall the example on net flux in Section 16.7

Suppose we want to compute the flux of $\mathbf{F} = (z - x)\mathbf{i} - \mathbf{j} + (z + 3)\mathbf{k}$ across the paraboloid S_1 with equation $z = 4 - x^2 - 4y^2$ for $z \geq 0$

The vector field $\mathbf{F}$ is incompressible, $\operatorname{div} \mathbf{F} = -1 + 1 = 0$. If we choose S_2 to be the elliptical disk $x^2 + 4y^2 \leq 4$, then we immediately have

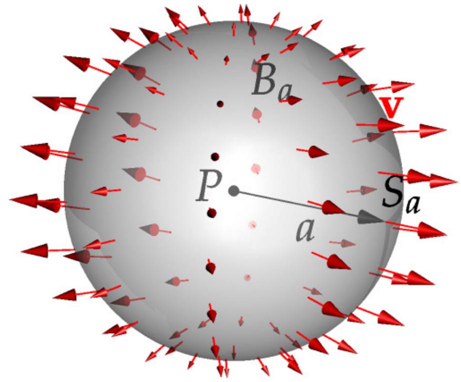


$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_1 &= -\iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}_2 \\
&= -\iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\,\mathrm{d}S_2 && \text{(orient } S_2 \textit{ downward}\text{)} \\
&= \iint_{S_2} z + 3\,\mathrm{d}S_2 = 3 \iint_{S_2} \mathrm{d}S_2 \\
&= 6\pi && \text{(area of ellipse)}
\end{aligned}$$

Interpreting Divergence

Let B_a be the solid ball with radius a, center P, and surface S_a. Let the vector field $\mathbf{v}$ satisfy the Divergence Theorem. Then

$$\iint_{S_a} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \iiint_{B_a} \nabla \cdot \mathbf{v} \, \mathrm{d}V = V_a (\nabla \cdot \mathbf{v})_{\text{av}}$$



where $(\nabla \cdot \mathbf{v})_{\text{av}}$ is the average divergence of $\mathbf{v}$ over B_a and $V_a = \frac{4}{3}\pi a^3$ is the volume of B_a. Therefore

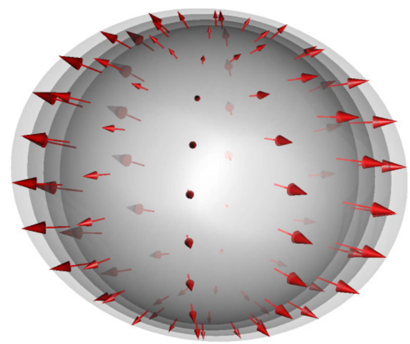
$$(\nabla \cdot \mathbf{v})_{\text{av}} = \frac{1}{V_a} \iint_{S_a} \mathbf{v} \cdot \mathrm{d}\mathbf{S}$$

Since $\mathbf{v}$ has continuous partial derivatives, we may take the limit as $a \to 0$ to obtain

$$(\nabla \cdot \mathbf{v})(P) = \lim_{a \to 0} \frac{1}{V_a} \iint_{S_a} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \lim_{a \to 0} \frac{1}{V_a} \iint_{S_a} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S$$

The divergence at P can therefore be thought of as the outward flux per unit volume.

Now let the surface S_a of the ball drift with the flow $\mathbf{v}$. Over time t, B_a approximately deforms to an ellipsoid with average radius $r_{\text{av}}(t)$. Clearly $\mathbf{v} \cdot \mathbf{n}_{\text{av}}$ is the average rate of change of the *radius* of B_a. The rate of change of volume is therefore



$$\left.\frac{\mathrm{d}V_a}{\mathrm{d}t}\right|_{t=0} \approx \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{4}{3}\pi r_{\text{av}}^3(t) = 4\pi r_{\text{av}}^2(0) r_{\text{av}}'(0) = 4\pi a^2 (\mathbf{v} \cdot \mathbf{n})_{\text{av}}$$

with improving approximation as $a \to 0$. It follows that

$$\nabla \cdot \mathbf{v}(P) = \lim_{a \to 0^+} \frac{1}{V_a} \iint_{S_a} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \lim_{a \to 0^+} \frac{1}{V_a} 4\pi a^2 (\mathbf{v} \cdot \mathbf{n})_{\text{av}} = \lim_{a \to 0^+} \frac{1}{V_a} \cdot \left.\frac{\mathrm{d}V_a}{\mathrm{d}t}\right|_{t=0} = \frac{1}{V} \frac{\mathrm{d}V}{\mathrm{d}t}$$

Divergence is therefore the (unitless) rate of change of volume.

Definition. *A point P at which $\nabla \cdot \mathbf{F}(P) > 0$ is called a* source *and a point at which $\nabla \cdot \mathbf{F}(P) < 0$ is called a* sink.

We can even interpret vector fields that are undefined this way, for example:
An inverse square field has the form $\mathbf{F} = r^{-3}\mathbf{r}$ for $r > 0$, where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Its divergence is easily seen to be zero[2] away from the origin:

$$\nabla \cdot \mathbf{F} = \nabla(r^{-3}) \cdot \mathbf{r} + r^{-3} \nabla \cdot \mathbf{r} = \frac{\frac{\mathrm{d}}{\mathrm{d}r} r^{-3}}{r} \mathbf{r} \cdot \mathbf{r} + r^{-3} \cdot 3 = -3r^{-5}r^2 + 3r^{-3} = 0$$

However, we could *define* the divergence of $\mathbf{F}$ at the origin using our above analysis:

$$\nabla \cdot \mathbf{F}(\mathbf{0}) := \lim_{a \to 0^+} \frac{1}{V_a} \iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \lim_{a \to 0^+} \frac{1}{V_a} \iint_{r=a} r^{-2} \, \mathrm{d}S = \lim_{a \to 0^+} \frac{3}{4\pi a^3} \cdot 4\pi a^2 a^{-2} = +\infty$$

We could therefore interpret $\mathbf{F}$ as having an infinite source at the origin.

[2]The product rule for divergence and how to calculate in spherical co-ordinates are in the homework.