16.10 Summary and Applications

Before seeing some applications of vector calculus to Physics, we note that vector calculus is *easy*, because...

There's only one Theorem!

Green's, Stokes', and the Divergence Theorem are all higher dimensional versions of the Fundamental Theorem of Calculus

More generally, there is a result known as Stokes' Theorem which is written

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha$$

M is a *manifold*[1], with boundary ∂M, α is a *differential form*[2] and $\mathrm{d}\alpha$ is its *exterior derivative*[3]

[1] A (possibly) multi-dimensional place where calculus can be done
[2] These generalize the concepts of function and vector field
[3] A generalization of div, grad and curl

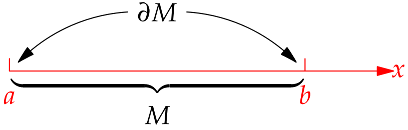
We have seen five examples of this very general theorem: in each case we need only interpret M, ∂M, α and $\mathrm{d}\alpha$

In each case α is a different object, but finding 'd' of it always involves taking a meaningful derivative[4]

1. $M = [a, b] \subset \mathbb{R}$ and $\alpha = f$, a function of one variable
   Here $\partial M = \{a, b\}$ and $\mathrm{d}\alpha = \mathrm{d}f = f'(x)\,\mathrm{d}x$
   The general Stokes' Theorem says

$$\int_{\partial M} f = f(b) - f(a) = \int_a^b f'(x)\,\mathrm{d}x = \int_M \mathrm{d}f,$$

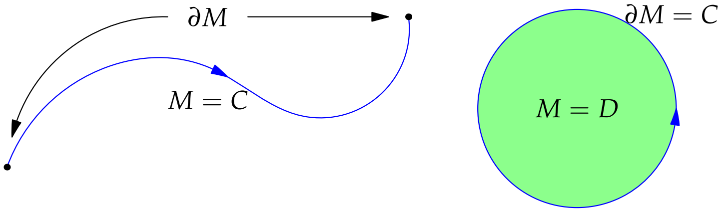
   which is simply the Fundamental Theorem of Calculus



[4]The meaning of 'd' will be made clear in an advanced Differential Geometry/Mathematical Physics (typically Relativity) course

2. $M = C$ a curve in $\mathbb{R}^n$ and $\alpha = f$, a function of n variables
∂M is the two endpoints of C and $\mathrm{d}\alpha = \mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r}$
We recover the Fundamental Theorem of Line Integrals:

$$\int_{\partial M} f = f(\mathrm{end}(C)) - f(\mathrm{start}(C)) = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = \int_M \mathrm{d}f$$

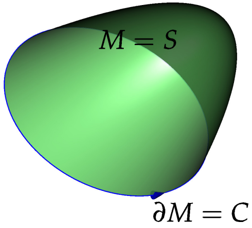


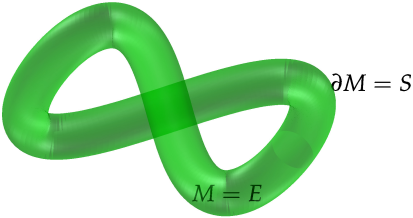
3. $M = D \subset \mathbb{R}^2$ with boundary curve $\partial M = C$
$\alpha = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
Here $\mathrm{d}\alpha = (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y$ and we get Green's Theorem:

$$\int_{\partial M} \alpha = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y = \int_M \mathrm{d}\alpha$$

4. $M = S \subset \mathbb{R}^3$ a surface with boundary curve $\partial M = C$
$\alpha = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
Here $\mathrm{d}\alpha = \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ and we obtain Stokes' Theorem:

$$\int_{\partial M} \alpha = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_M \mathrm{d}\alpha$$





5. $M = E \subset \mathbb{R}^3$ a volume with boundary surface $\partial M = S$
$\alpha = \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \mathbf{F} \cdot \mathrm{d}\mathbf{S}$
Here $\mathrm{d}\alpha = \nabla \cdot \mathbf{F}\,\mathrm{d}V$ from which we get the Divergence Theorem:

$$\int_{\partial M} \alpha = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}V = \int_M \mathrm{d}\alpha$$

Physical Applications

Vector Calculus is the language of Physics

1. Many famous equations are direct consequences of one of the Theorems of the course
2. Several equations are most conveniently written in terms of vector calculus

We apply the Divergence Theorem to the problems of heat and fluid flow and obtain fundamental equations

We also see Maxwell's Equations, which are usually written in the language of vector calculus

Finally we consider inverse square laws and Gauss' Laws for Gravitational and Electric fields

The Heat Equation

Let E be a solid[5] with temperature $T(\mathbf{r}, t)$ at position $\mathbf{r}$, time t
The heat energy leaving E per unit time is the flux integral

$$\iint_{\partial E} -k\nabla T \cdot \mathrm{d}\mathbf{S} = \iiint_E -k\nabla^2 T \,\mathrm{d}V$$

by the Divergence Theorem
Alternatively, the total heat energy in E is $\iiint_E \rho\sigma T \,\mathrm{d}V$, the rate of change of which is *negative* the heat flow out of E:

$$\iiint_E -k\nabla^2 T \,\mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t}\iiint_E \rho\sigma T \,\mathrm{d}V = -\iiint_E \rho\sigma\frac{\partial T}{\partial t}\,\mathrm{d}V$$

Since this holds for *any* volume E, the only possibility is that the integrands are equal and we get the heat equation:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho\sigma}\nabla^2 T$$

[5] With density ρ, conductivity k, and specific heat capacity σ

The Continuity Equation

Suppose a fluid has density $\rho(\mathbf{r}, t)$ and velocity field $\mathbf{v}$

The total mass of fluid in a region E is then $m = \iiint_E \rho \, \mathrm{d}V$

The rate of change of mass in E must be *negative* the flux integral across the boundary

$$\frac{\partial m}{\partial t} = -\iint_{\partial E} \rho \mathbf{v} \cdot \, \mathrm{d}\mathbf{S}$$

Applying the Divergence Theorem we obtain

$$\frac{\partial m}{\partial t} = \iiint_E \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\iint_{\partial E} \rho \mathbf{v} \cdot \, \mathrm{d}\mathbf{S} = -\iiint_E \nabla \cdot (\rho \mathbf{v}) \, \mathrm{d}V$$

and thus the *continuity equation*,[6] a fundamental equation in fluid mechanics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

[6]If density is constant the equation becomes $\nabla \cdot \mathbf{v} = 0$ giving physical meaning to the word *incompressible*

Maxwell's equations

These famous equations relate three vector fields: the *electric field* $\mathbf{E}$, the *magnetic field* $\mathbf{B}$, and the *current density* $\mathbf{J}$. The remaining terms are constants.[7]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss' Law for electricity)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss' Law for magnetism)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law of induction)}$$

$$\nabla \times \mathbf{B} = \frac{1}{\varepsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère's Law)}$$

[7] ε_0 is the permittivity of free space, c the speed of light, and ρ the charge density

Radial Vector Fields and Inverse Square Laws

Inverse square force laws pervade Physics:[8] why?

Start with three physical assumptions about a force field **F**:

1. Force acts in the direction of, or away from, its source
2. Magnitude depends only on distance away from source
3. Material should not accumulate: the net flux out of any region not containing the source should be zero

With the source at the origin, the first two conditions require that **F** is *radial*: there is some function f such that

$$\mathbf{F} = f(r)\frac{\mathbf{r}}{r}, \quad \text{where } r = |\mathbf{r}|, \text{ and } |\mathbf{F}| = |f(r)|$$

According to the Divergence Theorem, the third condition is equivalent to **F** being incompressible: i.e. $\operatorname{div} \mathbf{F} = 0$ for $r \neq 0$

I.e. there are no sources or sinks except at the origin

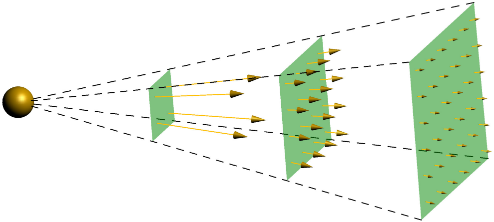
[8] E.g. Gravitation, Electromagnetism, etc…

If $\mathbf{F} = f(r)\frac{\mathbf{r}}{r}$ is to satisfy $\operatorname{div} \mathbf{F} = 0$, we compute:

$$\nabla \cdot \mathbf{F} = \nabla \left(\frac{f(r)}{r}\right) \cdot \mathbf{r} + \frac{f(r)}{r} \nabla \cdot \mathbf{r} = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{f(r)}{r}\right) \mathbf{r} \cdot \mathbf{r} + 3\frac{f(r)}{r}$$

$$= \frac{rf'(r) + 2f(r)}{r} = 0 \iff f(r) = kr^{-2}$$

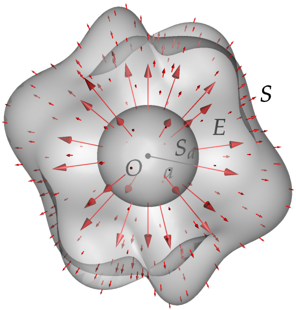
for some constant[9] $k \implies \mathbf{F}$ is an inverse square field



For example the intensity of light energy from the sun is an inverse square field: the flux (energy per second) across each of the green surfaces is identical

[9] $\Rightarrow$ direction needs a little Differential Equations…

Gauss' Law



We compare the fluxes of an inverse square field

$$\mathbf{F} = kr^{-3}\mathbf{r}$$

across a surface S surrounding the origin and a small sphere S_a of radius a inside S

Let E be the region between these surfaces so that $\partial E = S \cup S_a$

If both surfaces are oriented outward then, by the Divergence Theorem,

$$\left(\iint_S - \iint_{S_a}\right) \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, \mathrm{d}V = 0$$

Hence

$$\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \iint_{S_a} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}$$

It follows that

$$\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \iint_{S_a} kr^{-2}\frac{\mathbf{r}}{r} \cdot \, \mathrm{d}\mathbf{S} = \iint_{S_a} kr^{-2}\frac{\mathbf{r}}{r} \cdot \frac{\mathbf{r}}{r}\, \mathrm{d}S$$
$$= ka^{-2} \iint_{S_a} \mathrm{d}S = 4\pi k$$

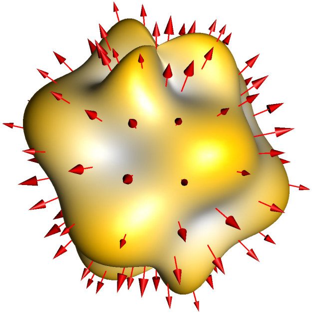
independently of the choice of sphere S_a and surface S

This result is called *Gauss' Law* for inverse square fields:

Theorem 16.10.1 (Gauss' Law)

The flux of an inverse square field $\mathbf{F} = kr^{-3}\mathbf{r}$ across any surface S enclosing the origin is

$$\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = 4\pi k$$



Gravitation

Gauss' Law makes it easy to consider the gravitational effect of several masses or a distribution of masses:

Suppose we have a distribution of density ρ ($\text{kg}\cdot\text{m}^{-3}$) spread over a region E and with total mass $M = \iiint_E \rho \, \mathrm{d}V$

The masses give rise to a gravitational force field $\mathbf{F}$

By Newton, an infinitessimal mass ΔM_i at $\mathbf{r}_i$ produces a gravitational force[10]

$$\Delta \mathbf{F}_i = \frac{-G\Delta M_i}{|\mathbf{r} - \mathbf{r}_i|^3}(\mathbf{r} - \mathbf{r}_i)$$

By Gauss, the total flux from all the masses is

$$\iint_{\partial E} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \sum \iint_{\partial E} \Delta \mathbf{F}_i \cdot \mathrm{d}\mathbf{S} = \sum -4\pi G \Delta M_i = -4\pi G M$$

[10] $G \approx 6.674 \times 10^{-11}\ \text{N}\cdot\text{m}^2\text{kg}^{-2}$ is the *gravitational constant*

By the Divergence Theorem we have

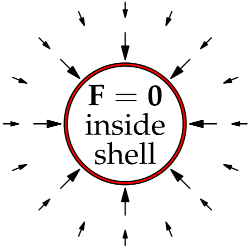
$$\iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}V = \iint_{\partial E} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -4\pi GM = \iiint_E -4\pi G\rho\,\mathrm{d}V$$

Since this holds for any region E, we conclude that

$$\nabla \cdot \mathbf{F} = -4\pi G\rho$$

This is known as *Gauss' Law of Gravitation*

Gauss' law makes the famous shell theorem obvious: a spherical shell of uniform density has no internal gravitational effect



Spherical mass distribution $\Longrightarrow \mathbf{F} = f(r)\frac{\mathbf{r}}{r}$ is spherically symmetric with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$

Then $\nabla \cdot \mathbf{F} = 0 \Longrightarrow f(r) = kr^{-2} \equiv 0$

Descent to the Center of the Earth. . .

Suppose that the density ρ of the Earth is constant[11]

If $\mathbf{F} = f(r)\frac{\mathbf{r}}{r}$ is the gravitational field at a distance r from the center of the earth, then Gauss' Law reads:

$$\nabla \cdot \mathbf{F} = f'(r) + \frac{2f(r)}{r} = \begin{cases} -4\pi G\rho & r < R \\ 0 & r > R \end{cases}$$

With a little differential equations, this can be solved to yield

$$f(r) = \begin{cases} -\frac{4}{3}\pi G\rho r & r < R \\ -\frac{4}{3}\pi G\rho R^3 r^{-2} = -GMr^{-2} & r > R \end{cases}$$

where R and M are the radius and mass of the Earth

Outside the Earth it is as if all the mass is at the center, but inside the force decreases in magnitude until, at the center, there is no gravitational effect

[11] It isn't, but it's not a dreadful assumption. There are many other `models`.

Charge Distribution and Faraday Cages

Gauss' Law also holds for charge: The electric field **E** generated by a distribution of charge of density ρ ($\mathrm{C}{\cdot}\mathrm{m}^{-3}$) satisfies[12]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

The analogue of the shell theorem says that an electric field external to a metal cage has no effect inside the cage: electrons on the surface will arrange themselves in such a way that $\mathbf{E} = 0$ inside

This technique is exploited by numerous `applications`, and helps explain why radios and cell phones don't work well in (almost) metal boxes like cars and train carriages

[12] $\varepsilon_0 \approx 8.854 \times 10^{12}\ \mathrm{F}{\cdot}\mathrm{m}^{-1}$ ($=\mathrm{C}{\cdot}\mathrm{V}^{-1}\mathrm{m}^{-1}$) is the *permittivity of free space*