Partial Differentiation To differentiate $f(x, y)$ with respect to x, treat y as a constant

Examples

1. Let $f(x, y) = 2x^2y^3 + \sin x$. Then
$$\frac{\partial f}{\partial x} = f_x(x, y) = 4xy^3 + \cos x, \quad \frac{\partial f}{\partial y} = f_y(x, y) = 6x^2y^2$$
2. Let $f(x, y, z) = x^2 + yz^{-1}$. Then
$$f_x(x, y, z) = 2x, \quad f_y(x, y, z) = z^{-1}, \quad f_z(x, y, z) = -yz^{-2}$$
3. Let $f(x, y, z) = x^2 \tan(yz^2)$. Then
$$f_x = 2x \tan(yz^2), \quad f_y = x^2z^2 \sec^2(yz^2), \quad f_z = 2x^2yz \sec^2(yz^2)$$

15 Multiple Integrals

15.2 Iterated Integrals

Interpretation: If R is a region in two-dimensions and f is an integrable function on R, then
$$\iint_R f \, \mathrm{d}A = f_{\mathrm{av}} \cdot \mathrm{Area}(R)$$
where f_{av} is the *average value* of the f over R. In particular $\iint_R 1 \, \mathrm{d}A = \mathrm{Area}(R)$

If E is the volume in three-dimensions *above* R and *underneath* the graph of $z = f(x, y)$, then
$$\iint_R f \, \mathrm{d}A = \mathrm{Volume}(E)$$
Otherwise said: Volume = Average height × Area of base

Theorem (Fubini). *Suppose $R = [a, b] \times [c, d]$ a rectangle and f continuous. Then*
$$\iint_R f(x, y) \, \mathrm{d}A = \int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_c^d \int_a^b f(x, y) \, \mathrm{d}x \, \mathrm{d}y$$
Order: evaluate inside integral first $\int_a^b \int_c^d f(x, y) \, \mathrm{d}y \, \mathrm{d}x$

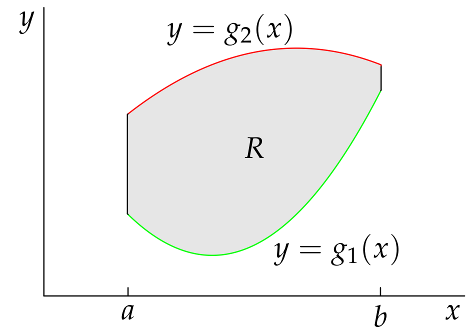
Example If $R = [1, 2] \times [0, 1]$ and $f(x, y) = x^2y$, then
$$\iint_R f(x, y) \, \mathrm{d}A = \int_0^1 \int_1^2 x^2y \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \left[\frac{1}{3}x^3y\right]_{x=1}^2 \mathrm{d}y = \int_0^1 \frac{7}{3}y \, \mathrm{d}y = \frac{7}{6}$$
This is easy for separable functions
$$\int_a^b \int_c^d g(x)h(y) \, \mathrm{d}y \, \mathrm{d}x = \int_a^b g(x) \, \mathrm{d}x \int_c^d h(y) \, \mathrm{d}y$$
Recalling the previous example: $f = gh$ where $g(x) = x^2$ and $h(y) = y$, whence
$$\iint_R f(x, y) \, \mathrm{d}A = \int_0^1 \int_1^2 x^2y \, \mathrm{d}x \, \mathrm{d}y = \int_1^2 x^2 \, \mathrm{d}x \int_0^1 y \, \mathrm{d}y = \frac{7}{3} \cdot \frac{1}{2} = \frac{7}{6}$$

15.3 Double Integrals over General Regions

Type 1: $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$

Integrate with respect to y first

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$



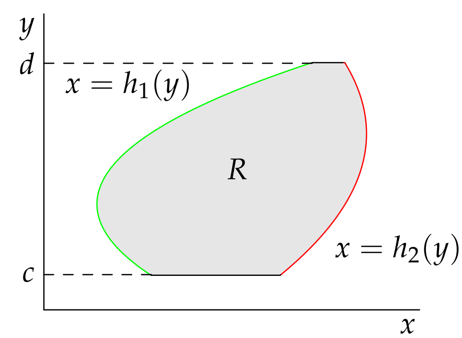
Example For R defined by $0 \le x \le 1$ and $3x \le y \le 8 - 4x^2$ we have

$$\iint_R x\,\mathrm{d}A = \int_0^1 \int_{3x}^{8-4x^2} x\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 xy\Big|_{y=3x}^{y=8-4x^2}\mathrm{d}x = \int_0^1 8x - 4x^3 - 3x^2\,\mathrm{d}x$$
$$= 4x^2 - x^4 - x^3\Big|_0^1 = 2$$

Type 2: $h_1(y) \le x \le h_2(y)$ and $c \le y \le d$

Integrate with respect to x first

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$



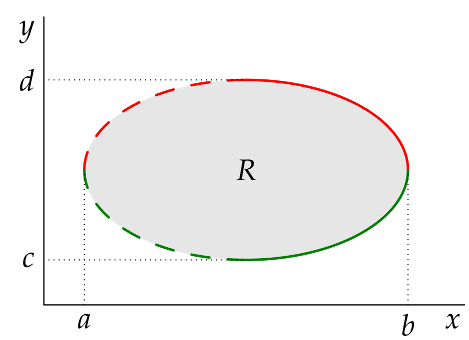
Example For R defined by $0 \le y \le 1$ and $y \le x \le e^y$ we have

$$\iint_R 2x\,\mathrm{d}A = \int_0^1 \int_y^{e^y} 2x\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 x^2\Big|_{x=y}^{x=e^y}\mathrm{d}y = \int_0^1 e^{2y} - y^2\,\mathrm{d}y = \frac{1}{2}e^{2y} - \frac{1}{3}y^3\Big|_0^1$$
$$= \frac{1}{2}e^2 - \frac{1}{3} - \frac{1}{2} = \frac{1}{2}e^2 - \frac{5}{6}$$

Both Type 1 + Type 2: Integrate either way!

R can be described

$$\begin{cases} a \le x \le b \text{ and } g_1(x) \le y \le g_2(x), & \text{or} \\ c \le y \le d \text{ and } h_1(y) \le x \le h_2(y) \end{cases}$$



Example Triangle T is a region of type 1

$$0 \le x \le 1, \quad 0 \le y \le 3 - 3x$$

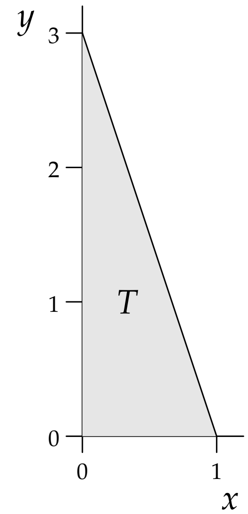
and type 2

$$0 \le y \le 3, \quad 0 \le x \le 1 - \frac{y}{3}$$

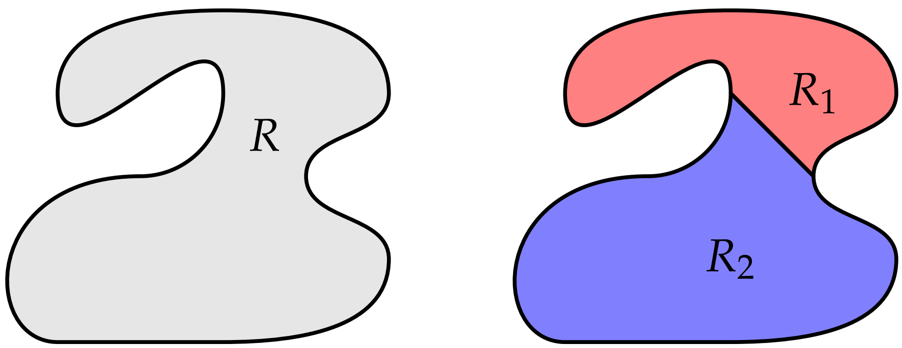
Hence

$$\iint_T x \, \mathrm{d}A = \int_0^1 \int_0^{3-3x} x \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 3x - 3x^2 \, \mathrm{d}x = \frac{1}{2}$$

$$\text{Or} = \int_0^3 \int_0^{1-\frac{y}{3}} x \, \mathrm{d}x \, \mathrm{d}y = \int_0^3 \frac{1}{2}\left(1 - \frac{y}{3}\right)^2 \mathrm{d}y = \frac{1}{2}$$



Other regions: Cut region two create several integrals of either type. For example the following region may be sub-divided into two regions of type 1



For any function f, we have $\iint_R f \, \mathrm{d}A = \iint_{R_1} f \, \mathrm{d}A + \iint_{R_2} f \, \mathrm{d}A$

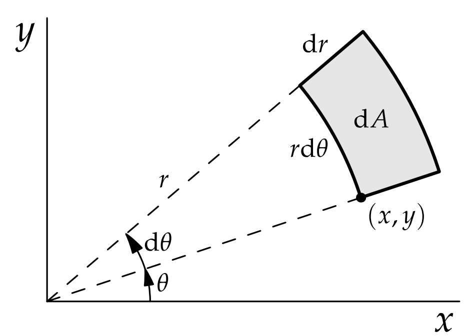
15.4 Double Integrals in Polar Co-ordinates

Polar co-ordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \leftrightsquigarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$

Infinitessimal Area: Starting at (x, y), increase polar co-ordinates by infinitessimal amounts $\mathrm{d}r$ and $\mathrm{d}\theta$. Infinitessimal area is swept out:[a]

$$\mathrm{d}A = \left(\pi(r+\mathrm{d}r)^2 - \pi r^2\right)\frac{\mathrm{d}\theta}{2\pi} = r \, \mathrm{d}r \, \mathrm{d}\theta$$

since $(\mathrm{d}r)^2 \ll \mathrm{d}r$ for infinitessimals.

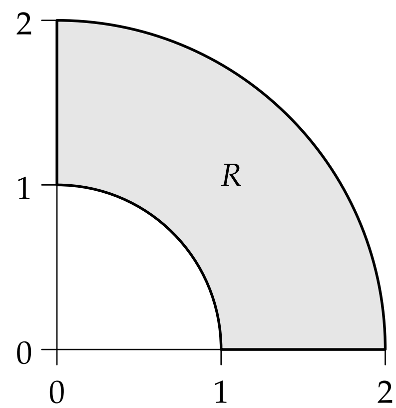


[a] $\mathrm{d}A$ is the area of a segment between two circles

Theorem. *Suppose that R is a polar rectangle defined by $r_1 \le r \le r_2$ and $\theta_1 \le \theta \le \theta_2$ and that f is a continuous function of R. Then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

Example Find $\iint_R 4x+3\,\mathrm{d}A$ for the annular region R described by $1 \le x^2+y^2 \le 4$ with $x,y \ge 0$



In polar co-ordinates R is $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$. Hence

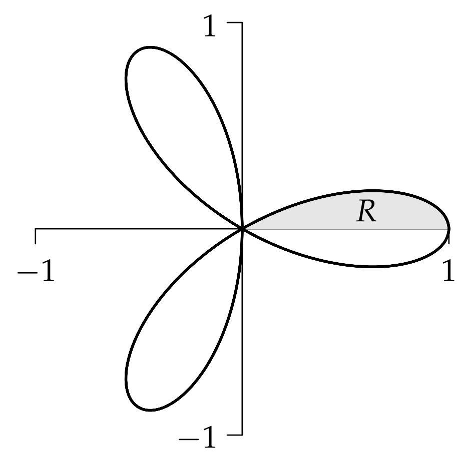
$$\begin{aligned}\iint_R 4x+3\,\mathrm{d}A &= \int_1^2\int_0^{\frac{\pi}{2}} 4r^2\cos\theta + 3r\,\mathrm{d}\theta\,\mathrm{d}r\\ &= \int_1^2 4r^2\sin\theta + 3\theta r\Big|_{\theta=0}^{\pi/2}\mathrm{d}r\\ &= \int_1^2 4r^2 + \frac{3\pi}{2}r\,\mathrm{d}r = \frac{28}{3}+\frac{9\pi}{4}\end{aligned}$$

Advanced: The Theorem may be modified for regions of the plane where $g(\theta) \le r \le h(\theta)$ for some functions of g,h of r (the polar equivalent of Type 1):

$$\iint_R f\,\mathrm{d}A = \int_{\theta_1}^{\theta_2}\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

A similarly approach can be taken for the analogue of a region of Type 2.

Example The three-leaved rose has equation $r = \cos 3\theta$. Find its area.



Choose R to be half of one leaf: $0 \le r \le \cos 3\theta$ and $0 \le \theta \le \frac{\pi}{6}$

$$\begin{aligned}\iint_R \mathrm{d}A &= \int_0^{\pi/6}\int_0^{\cos 3\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi/6}\frac{1}{2}r^2\Big|_{r=0}^{\cos 3\theta}\mathrm{d}\theta\\ &= \int_0^{\pi/6}\frac{1}{2}\cos^2 3\theta\,\mathrm{d}\theta = \frac{1}{4}\int_0^{\pi/6}\cos 6\theta + 1\,\mathrm{d}\theta\\ &= \frac{1}{4}\left[\frac{1}{6}\sin 6\theta + \theta\right]_0^{\pi/6} = \frac{\pi}{24}\end{aligned}$$

By symmetry, the total area of the rose is therefore $6\cdot\frac{\pi}{24} = \frac{\pi}{4}$

15.7 Triple Integrals

Interpretation: If E is a region in two-dimensions and f is an integrable function on E, then

$$\iiint_E f\,\mathrm{d}V = f_{\text{av}} \cdot \text{Volume}(E)$$

where f_{av} is the *average value* of the f over E. In particular $\iiint_E 1\,\mathrm{d}V = \text{Volume}(E)$

For example, if $T(x,y,z)$ is the temperature at a point (x,y,z) in a room E, then the average temperature in the room is

$$T_{\text{av}} = \frac{1}{\text{Volume}(E)} \iiint_E f\,\mathrm{d}V$$

$\iiint_E f\,\mathrm{d}V$ can be interpreted as a *hypervolume* in four-dimensions, but this is unhelpful to most of us!

Theorem (Fubini). *Suppose that $E = [p,q] \times [r,s] \times [t,u]$ is a cuboid and f is continuous on E. Then*

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_p^q \int_r^s \int_t^u f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

More generally, if E is the region defined by the inequalities

$$\begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \\ h_1(x,y) \le z \le h_2(x,y) \end{cases}$$

then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

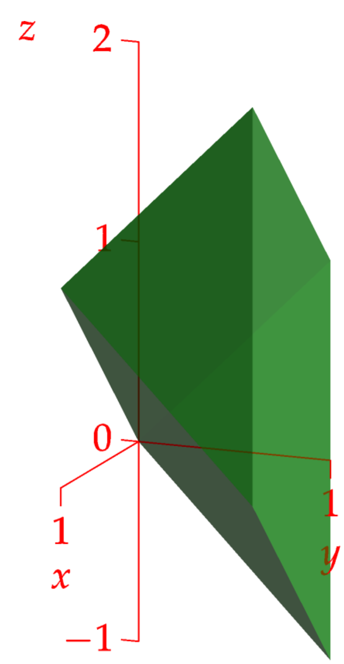
In general there are *six* ways of ordering the variables x, y, z.

Example Find the integral $\iiint_V f\,\mathrm{d}V$, where

$$f(x,y,z) = x + 2yz$$

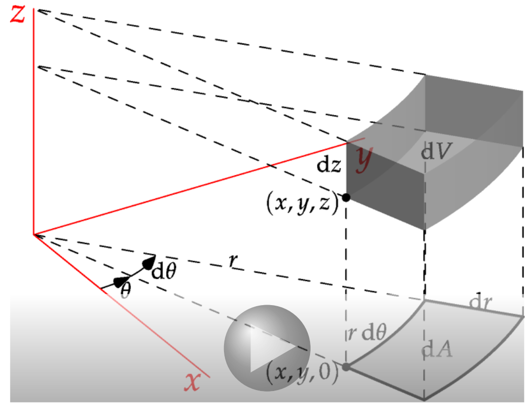
and V is defined by

$$0 \le x, y \le 1,$$
$$x - y \le z \le x + y$$



$$\begin{aligned} \iiint_V f\,\mathrm{d}V &= \int_0^1 \int_0^1 \int_{x-y}^{x+y} x + 2yz\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \int_0^1 xz + yz^2 \Big|_{z=x-y}^{x+y} \mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \int_0^1 x(x+y-(x-y)) + y((x+y)^2 - (x-y)^2)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \int_0^1 2xy + 4xy^2\,\mathrm{d}y\,\mathrm{d}x = \frac{7}{6} \end{aligned}$$

15.8 Triple Integrals in Cylindrical Co-ordinates

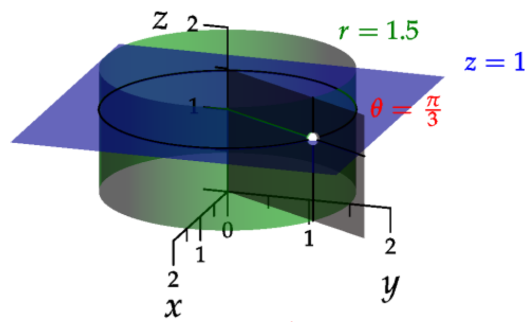


Polar co-ordinates + z

$$\mathrm{d}V = \mathrm{d}A\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

Useful when the domain of integration has rotational symmetry, or when x^2+y^2 is dominant in the integrand

$$\iiint_E f\,\mathrm{d}V = \iiint_E f(r\cos\theta, r\sin\theta, z)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

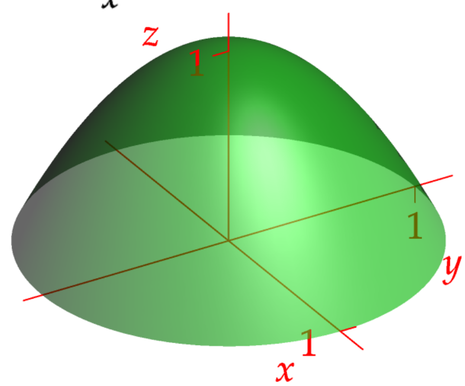


Co-ordinate surfaces

Constant z: horizontal planes

Constant r: cylinders

Constant θ: planes touching z-axis



Example Calculate the integral of the function

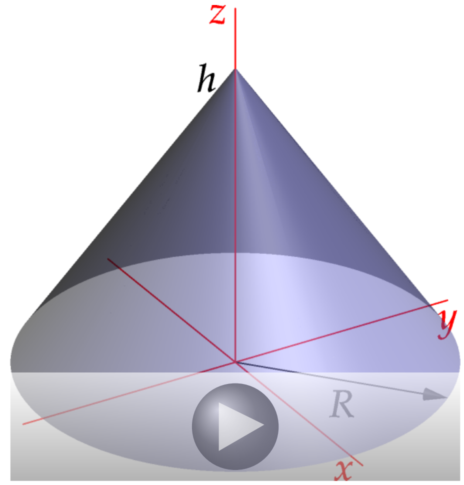
$$f(x,y,z) = x^2+y^2+2z$$

under the paraboloidal cap $z = 1-x^2-y^2$ and above the xy-plane.

In cylindrical polars, the cap has equation $z = 1-r^2$, and intersects the plane $z=0$ in the circle $r=1$, hence

$$\iiint_V f\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2}(2z+r^2)r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= 2\pi\int_0^1 rz^2+r^3z\Big|_{z=0}^{1-r^2}\mathrm{d}r = 2\pi\int_0^1 r(1-r^2)^2+r^3-r^5\,\mathrm{d}r$$
$$= 2\pi\left[-\frac{1}{6}(1-r^2)^3+\frac{1}{4}r^4-\frac{1}{6}r^6\right]_0^1 = \frac{\pi}{2}$$

Example A cone has height h and circular base of radius R. Find its volume using an integral.



The cone is created by rotating the line joining $(0, 0, h)$ and $(R, 0, 0)$ around the z-axis. This line (in the xz-plane) has equation

$$\frac{z}{h} + \frac{x}{R} = 1$$

Rotating this simply means replacing x with the radius, this the cone has equation

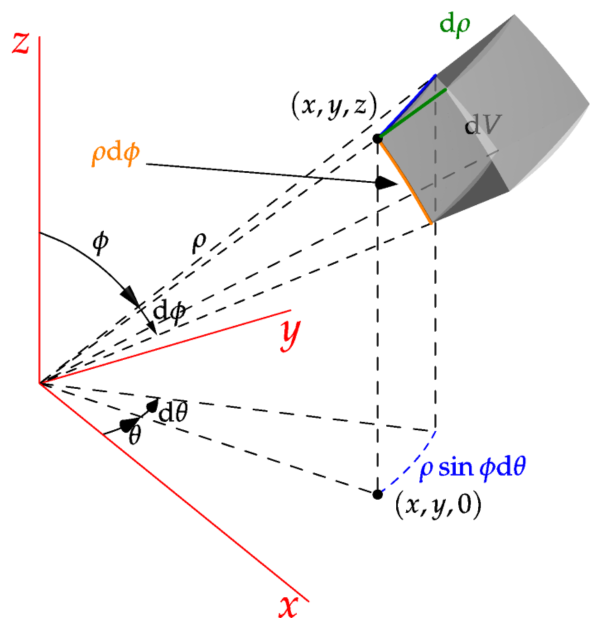
$$\frac{z}{h} + \frac{r}{R} = 1 \implies z = h\left(1 - \frac{r}{R}\right)$$

Its volume is therefore

$$\iiint_V \mathrm{d}V = \int_0^{2\pi} \int_0^R \int_0^{h\left(1-\frac{r}{R}\right)} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta = 2\pi \int_0^R rh\left(1 - \frac{r}{R}\right) \mathrm{d}r$$
$$= 2\pi h \int_0^R r - \frac{r^2}{R} \,\mathrm{d}r = 2\pi h \left(\frac{R^2}{2} - \frac{R^3}{3R}\right) = \frac{1}{3}\pi h R^2$$

15.9 Triple Integrals in Spherical Co-ordinates

Three co-ordinates:
ρ: the distance from the origin
ϕ: the angle down from the positive z-axis
θ: the polar angle in the xy-plane



$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

$$\rho \geq 0 \qquad 0 \leq \phi \leq \pi \qquad 0 \leq \theta < 2\pi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{y}{x} \qquad \cos\phi = \frac{z}{\rho}$$

Compute infinitessimal volume by increasing each co-ordinate by small amount: volume swept out is approximately cuboidal, with volume

$$\mathrm{d}V = \mathrm{d}\rho \,\rho \sin\phi \,\mathrm{d}\theta \,\rho \,\mathrm{d}\phi = \rho^2 \sin\phi \,\mathrm{d}\rho \,\mathrm{d}\theta \,\mathrm{d}\phi$$

Warning! In many places later on, r is used instead of ρ: make sure you know which co-ordinate system (cylindrical or spherical) you are using!

Example $(x, y, z) = (1, \sqrt{3}, 3)$ has spherical polar co-ordinates

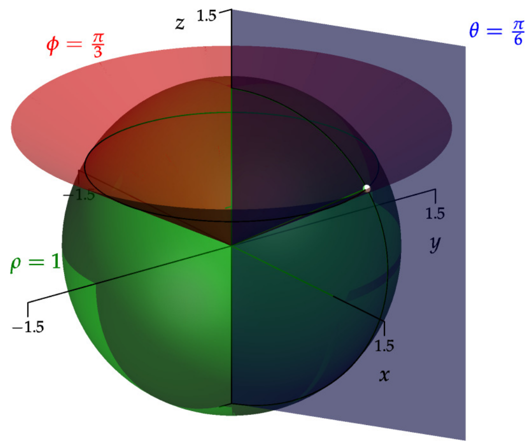
$$(\rho, \phi, \theta) = \left(\sqrt{13}, \cos^{-1} \frac{3}{\sqrt{13}}, \frac{\pi}{3}\right)$$

Co-ordinate surfaces

ρ constant: sphere radius ρ

θ constant: plane touching z-axis making angle θ with xz-plane

ϕ constant: cone centered on z-axis, angle ϕ from vertical



Example A sphere of radius R has volume

$$\int_0^{\pi} \int_0^{2\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^{R} \rho^2 \, d\rho$$
$$= 2 \cdot 2\pi \cdot \frac{1}{3} R^3 = \frac{4}{3}\pi R^3$$

Example A solid lies above the cone

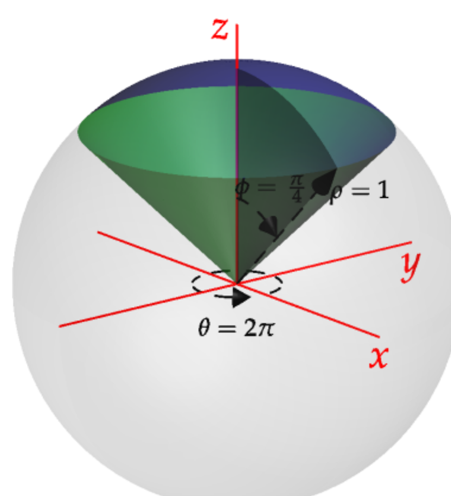
$$z = \sqrt{x^2 + y^2}$$

and below the sphere

$$x^2 + y^2 + z^2 = 1$$

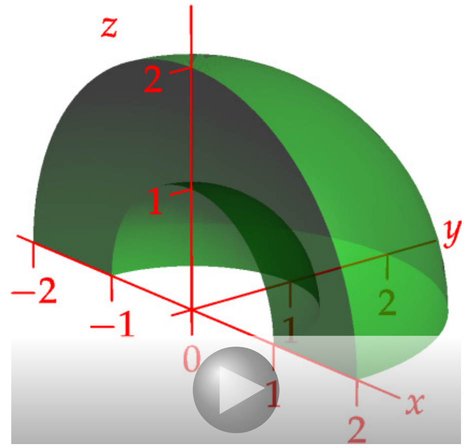
Its volume is

$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = (-\cos\phi)\Big|_0^{\pi/4} \cdot 2\pi \cdot \frac{1}{3}\rho^3\Big|_0^1$$
$$= \left(1 - \frac{1}{\sqrt{2}}\right) \cdot 2\pi \cdot \frac{1}{3} = \frac{(2 - \sqrt{2})\pi}{3}$$



Example Find the integral of $f(x, y, z) = yz$ over the volume shown

$$1 \le \rho \le 2, \qquad 0 \le \theta \le \pi$$
$$0 \le \phi \le \frac{\pi}{2}$$



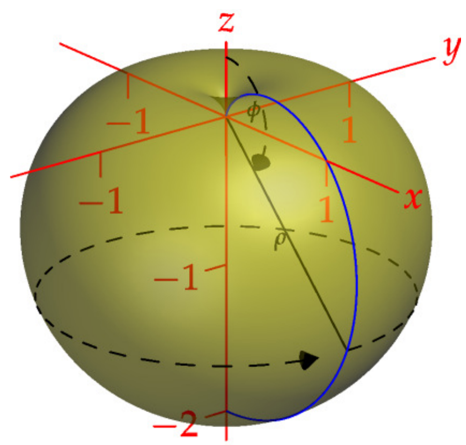
$$\begin{aligned}\iiint_V f\,\mathrm{d}V &= \int_1^2\int_0^{\pi/2}\int_0^{\pi} \rho\sin\phi\sin\theta\,\rho\cos\phi\,\rho^2\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\rho \\ &= \int_1^2 \rho^4\,\mathrm{d}\rho \cdot \int_0^{\pi/2} \cos\phi\sin^2\phi\,\mathrm{d}\phi \cdot \int_0^{\pi} \sin\theta\,\mathrm{d}\theta \\ &= \frac{1}{5}(32-1)\cdot\frac{1}{3}\sin^3\phi\Big|_0^{\pi/2}\cdot 2 = \frac{62}{15}\end{aligned}$$

Example A plum is modeled by the equation

$$\rho = 1 - \cos\phi$$

If ρ is measured in inches, find the volume of the plum

If you think back to 2D, $r = 1 - \cos\theta$ is the equation of a *cardioid* in polar co-ordinates. The plum is just the surface formed by rotating a cardioid.



$$\begin{aligned}\text{Volume} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= 2\pi\int_0^{\pi}\frac{1}{3}(1-\cos\phi)^3\sin\phi\,\mathrm{d}\phi \\ &= 2\pi\cdot\frac{1}{12}(1-\cos\phi)^4\Big|_0^{\pi} = 2\pi\cdot\frac{16}{12} = \frac{8\pi}{3}\ \text{in}^3\end{aligned}$$

For a sanity check, this is precisely the volume of a sphere of radius $\sqrt[3]{2} \approx 1.2599$ in.

15.10 Change of Variables in Multiple Integrals

How to integrate in arbitrary co-ordinates?

Definition. *Let* $(x, y) = (x(u, v), y(u, v))$ *be a transformation of co-ordinates*
The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} := \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = z\text{-component of } \begin{pmatrix} x_u \\ y_u \\ 0 \end{pmatrix} \times \begin{pmatrix} x_v \\ y_v \\ 0 \end{pmatrix}$$

Example Suppose $x = u - 2v$ and $y = 3u + 4v$. Then

$$\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u = 1\cdot 4 - (-2)\cdot 3 = 10$$

Theorem. *The Jacobian of the inverse transform* $(u, v) = (u(x, y), v(x, y))$ *is*

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1}$$

Example If $x = u - 2v$ and $y = 3u + 4v$, then we may solve for u, v to obtain

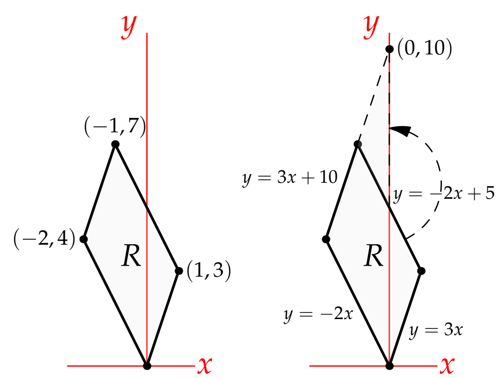
$$u = \frac{1}{5}(2x + y) \quad \text{and} \quad v = \frac{1}{10}(y - 3x)$$

The inverse Jacobian is therefore

$$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x = \frac{2}{5}\cdot\frac{1}{10} - \frac{1}{5}\cdot\frac{-3}{10} = \frac{1}{10}$$

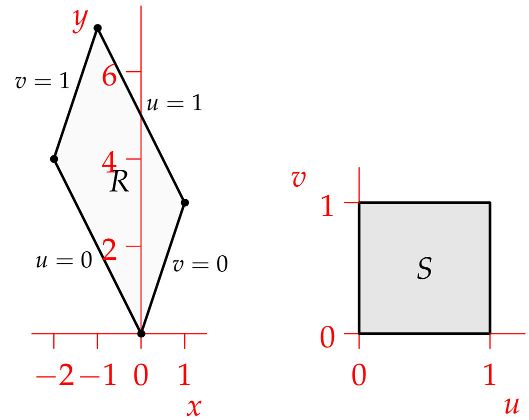
What does this have to do with integration? Let the parallelogram R have corners $(0, 0)$, $(1, 3)$, $(-1, 7)$ and $(-2, 4)$ It is easy to see that its area is $\frac{1}{2}\cdot 10\cdot 2 = 10$

Opposite edges of the parallelogram are parallel lines, the equations of which are similar.
For example, $y = -2x$ and $y = -2x + 5$ may be written $y + 2x = 0$ and $y + 2x = 5$: on opposite edges, the *same* function of x and y is equal to two *different constants.*



If we *define* the functions $u = \frac{1}{5}(2x + y)$ and $v = \frac{1}{10}(y - 3x)$, then the four edges of the parallelogram may be described as $u = 0, u = 1, v = 0, v = 1$. With respect to the *new co-ordinates* (u, v), the parallelogram becomes a *square* S with area 1.

The factor relating the (u, v)-area and the (x, y)-area is precisely the Jacobian:



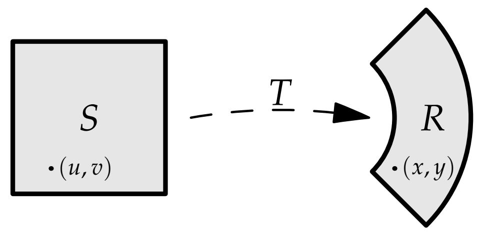
$$\text{Area}_{(x,y)} = 10\text{Area}_{(u,v)} = \frac{\partial(x, y)}{\partial(u, v)}\text{Area}_{(u,v)}$$

Written in terms of a double integral, this reads

$$\iint_R \mathrm{d}A = \iint_R \mathrm{d}x\,\mathrm{d}y = \iint_S \frac{\partial(x, y)}{\partial(u, v)}\,\mathrm{d}u\,\mathrm{d}v$$

The same idea applies to any change of co-ordinates. Given a domain of integration R, search for functions $u(x, y), v(x, y)$ so that, in terms of u, v, the domain becomes a simpler shape S, one over which integration is simpler.

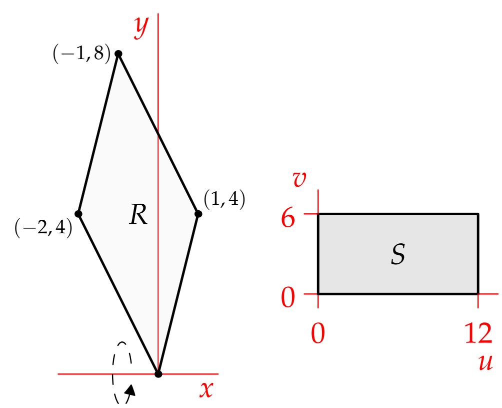
Theorem. *Suppose S is a region in the (u, v)-plane that is mapped 1–1 onto a region R in the (x, y)-plane by a transformation $(x, y) = T(u, v)$ with continuous 1st partial derivatives. If f is a continuous function on S, then*



$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_S f(T(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$$

Warning! Take the *absolute value* of the Jacobian. This was not needed in our parallelogram example since the Jacobian was already positive.

Example Find the moment of inertia $\iint_R y^2\,\mathrm{d}A$ of the parallelogram R about the x-axis



First find the equations of the edges: $y = 4x, y = 4x + 12$ and $y = -2x, y = -2x + 6$. This suggests new co-ordinates

$$u = y - 4x, \qquad v = y + 2x$$

R becomes the rectangle S defined by $0 \le u \le 12$, $0 \le v \le 6$.

Compute the Jacobian:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 2 & 1 \end{vmatrix} = -6$$

Finally, solve for y to transform the integrand, $u + 2v = 3y \implies y = \frac{1}{3}(u + 2v)$, and compute:

$$\iint_R y^2\,\mathrm{d}x\,\mathrm{d}y = \iint_S \frac{(u+2v)^2}{3^2}\left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v = \int_0^6 \int_0^{12} \frac{(u+2v)^2}{3^2} \cdot \left|\frac{1}{-6}\right| \mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{54}\int_0^6 \int_0^{12} (u + 2v)^2\,\mathrm{d}u\,\mathrm{d}v = 224$$

Polar co-ordinates The usual formula for converting an integral to polar co-ordinates has the same Jacobian origin: $x = r\cos\theta$, $y = r\sin\theta$ have Jacobian

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\implies \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_S f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta$$

We can also convert the other way: $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$ yields

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}}$$

$$\implies \iint_R f(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta = \iint_S f\left(\sqrt{x^2+y^2}, \tan^{-1}\frac{y}{x}\right)\frac{1}{\sqrt{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

Change of Variables in Triple Integration The idea is identical to that for double integrals, we simply need a Jacobian for three variables.

Definition. *Let $(x,y,z) = (x(u,v,w), y(u,v,w), z(u,v,w))$ be a transformation of co-ordinates The Jacobian of the transformation is*

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \cdot \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} \times \begin{pmatrix} x_w \\ y_w \\ z_w \end{pmatrix}$$

Example If $x = v - w$, $y = -u + w$, and $z = u - 2v$, then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = -1$$

Theorem. *Suppose S is region of (u,v,w)-space mapped 1–1 onto a region R in (x,y,z)-space by a transformation $(x,y,z) = T(u,v,w)$ with continuous 1st partial derivatives. If f is continuous on R, then*

$$\iiint_R f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_S f(T(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Just as for polar co-ordinates, we can use this method to derive other change of variable formulæ:

Cylindrical Polar Co-ordinates $x = r\cos\theta, y = r\sin\theta, z = z$ gives

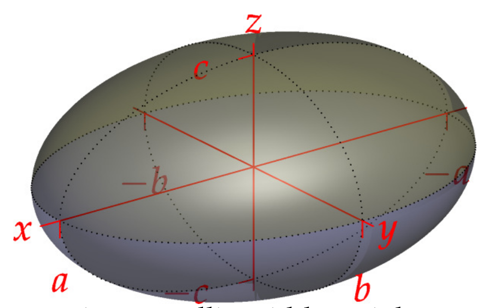
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \implies \mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

Spherical Polar Co-ordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{pmatrix}$ gives

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \cdot \begin{pmatrix} \rho^2\sin^2\phi\cos\theta \\ \rho^2\sin^2\phi\sin\theta \\ \rho^2\sin\phi\cos\phi \end{pmatrix} \\
&= \rho^2\sin\phi(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi) \\
&= \rho^2\sin\phi
\end{aligned}$$

Since $\sin\phi \geq 0$ for the allowed range of spherical co-ordinates ($0 \leq \phi \leq \pi$), we conclude that

$$\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$



The Volume of an Ellipsoid Finally, consider applying a change of co-ordinates to compute the volume of a general ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

We could employ a brute force approach in Cartesian co-ordinates: since an ellipsoid has eight octants each with the same volume, we could compute

$$V = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - x^2/a^2 - y^2/b^2}\,\mathrm{d}y\,\mathrm{d}x$$

Continuing from here requires a tricky trig substitution: let $y = b\sqrt{1-x^2/a^2}\sin\theta$ so that $\mathrm{d}y = b\sqrt{1-x^2/a^2}\cos\theta\,\mathrm{d}\theta$ and

$$V = 8bc\int_0^a \int_0^{\pi/2} (1 - x^2/a^2)\cos^2\theta\,\mathrm{d}\theta\,\mathrm{d}x = \cdots = \frac{4}{3}\pi abc$$

A much simpler approach involves changing co-ordinates so that the ellipsoid becomes a sphere. Since we know the volume of a sphere, the calculation becomes trivial.
Let $x = au$, $y = bv$, $z = cw$, then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = abc$$

In u, v, w co-ordinates, the ellipsoid has equation $u^2 + v^2 + w^2 = 1$: it has become a unit sphere! If E is the solid ellipsoid and S the solid sphere, then

$$V = \iiint_E \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_S abc\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = abc \cdot \text{Volume}(S) = \frac{4}{3}\pi abc$$