Math 2E Suggested Written Questions 16.1, 16.2

16.1 Vector Fields

1. Find the gradient field of the function $f(x, y) = \tan(3x - 4y)$.
2. Find the gradient field of the function $f(x, y, z) = x \ln(y - 2z)$.
3. Find the gradient field ∇f of $f(x, y) = \sqrt{x^2 + y^2}$ and sketch it.
4. At time $t = 1$ a particle is located at position (1,3). If it moves in a velocity field

$$\mathbf{F}(x, y) = \begin{pmatrix} xy - 2 \\ y^2 - 10 \end{pmatrix}$$

   find its approximate location at time $t = 1.05$.

5. (Non-examinable: for those who have taken Math 3D) The *flow lines* (or *streamlines*) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.
   (a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?
   (b) If the parametric equations of a flow line are $x = x(t), y = y(t)$, explain why these functions must satisfy the differential equations $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = -y$. Solve the differential equations to find an equation of the flow line that passes through the point (1,1).
6. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ and then sketch some flow lines. What shape do these lines appear to have?
   (b) If the parametric equations of a flow line are $x = x(t), y = y(t)$, what differential equations must these functions satisfy? Deduce that $\frac{dy}{dx} = x$.
   (c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

16.2 Line Integrals

1. Evaluate the line integral $\int_C x \sin y \, ds$ where C is the line segment from $(0, 3)$ to $(4, 6)$.
2. Evaluate the line integral $\int_C x^2 \, dx + y^2 \, dy$ where C consists of the quarter-arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ followed by the line segment from $(0, 2)$ to $(4, 3)$.
3. Evaluate the line integral $\int_C xyz^2 \, ds$ where C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.
4. Evaluate the line integral $\int_C y \, dx + z \, dy + x \, dz$ where C is parametrized by

$$x = \sqrt{t}, \quad y = t, \quad z = t^2, \quad \text{for } 1 \leq t \leq 4$$

5. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y - z)\mathbf{j} + z^2\mathbf{k}$, and C is parametrized by $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$ for $0 \leq t \leq 1$.

6. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$, and C is parametrized by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ for $0 \le t \le \pi$.

7. Use a graph of the vector field $\mathbf{F}(x, y) = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix}$ and the curve C, the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$, to guess whether the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative or zero. Then evaluate the line integral.

8. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented counter-clockwise.
   (b) Sketch a graph of the force field and the circle. Use it to explain your answer to part (a).

9. Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + ye^x\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

10. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is

$$\mathbf{F}(\mathbf{r}) = \frac{K\mathbf{r}}{|\mathbf{r}|^3}$$

where K is a constant. Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

11. An object with mass m moves with position function $\mathbf{r}(t) = a \sin t\mathbf{i} + b \cos t\mathbf{j} + ct\mathbf{k}$ for $0 \le t \le \frac{\pi}{2}$. Find the work done (by gravity) on the object during this time period.

12. The base of a circular fence of radius 10 m is given by $x = 10 \cos t,\ y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 liter of paint covers 100 m^2. Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

13. If C is a smooth curve given by a vector function $\mathbf{r}(t)$ for $a \le t \le b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]$$