Math 2E Suggested Written Questions 16.3

16.3 The Fundamental Theorem for Line Integrals

1. Determine whether $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.
   (a) $\mathbf{F}(x, y) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$ on the plane $\mathbb{R}^2$.
   (b) $\mathbf{F}(x, y) = (3x^2 - 2y^2)\,\mathbf{i} + (4xy + 3)\,\mathbf{j}$ on the plane $\mathbb{R}^2$.
   (c) $\mathbf{F}(x, y) = (2xy + y^{-2})\,\mathbf{i} + (x^2 - 2xy^{-3})\,\mathbf{j}$, on the half-plane where $y > 0$.
2. First find a function f such that $\mathbf{F} = \nabla f$, then use it to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along the given curve C.
   (a) $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$. C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.
   (b) $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$. C: $x = \sqrt{t},\ y = t + 1,\ z = t^2,\ 0 \le t \le 1$.
   (c) $\mathbf{F}(x, y, z) = \sin y\mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z\mathbf{k}$. C: $\mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le \frac{\pi}{2}$.
3. Show that the integral $\int_C \sin y\,\mathrm{d}x + (x \cos y - \sin y)\,\mathrm{d}y$ is independent of path and evaluate it if C is any path from $(2, 0)$ to $(1, \pi)$.
4. Suppose an experiment determines that the amount of work required for a force field $\mathbf{F}$ to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by $\mathbf{F}$ in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about $\mathbf{F}$? Why?
5. Find the work done by the force field $\mathbf{F} = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$ in moving a particle from $(0, 1)$ to $(2, 0)$.
6. Show that the line integral $\int_C y\,\mathrm{d}x + x\,\mathrm{d}y + xyz\,\mathrm{d}z$ is not independent of path.
7. Determine whether the given set is: open, connected, simply-connected.
   (a) $\{(x, y) : 1 < |x| < 2\}$
   (b) $\{(x, y) : (x, y) \ne (2, 3)\}$
8. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,
   $$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3} = \frac{c}{r^3}\mathbf{r}$$
   for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$. Find the work done in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.
   (b) An example of an inverse square field is the gravitational field $\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}$. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^{11} m from the sun) to perihelion (at a minimum distance of 1.47×10^{11} m). Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ $\mathrm{Nm}^2/\mathrm{kg}^2$.
   (c) Another example of an inverse square field is the electric force field $\mathbf{F} = \frac{\varepsilon q Q}{r^3}\mathbf{r}$. Suppose that an electron with a charge of $Q = -1.6 \times 10^{-19}$ C is located at the origin. A proton with charge $q = 1.6 \times 10^{-19}$ C is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. Use the value $\varepsilon = 8.985 \times 10^9$ $\mathrm{Nm}^2/\mathrm{C}^2$.