

Math 2E Suggested Written Questions 16.4, 16.5

16.4 Green's Theorem

- Evaluate the line integral directly and then using Green's Theorem.
 - $\oint_C xy \, dx + x^2 \, dy$. C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,1)$, and $(0,1)$.
 - $\oint_C x^2 y^2 \, dx + xy \, dy$. C consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.
- Use Green's Theorem to evaluate the line integral along the given positively oriented curve.
 - $\oint_C \cos y \, dx + x^2 \sin y \, dy$. C is the rectangle with vertices $(0,0)$, $(5,0)$, $(5,2)$, and $(0,2)$.
 - $\oint_C (1 - y^3) \, dx + (x^3 + e^{y^2}) \, dy$. C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
- Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Check the orientation of the curve before applying the theorem.
 - $\mathbf{F}(x,y) = (e^{-x} + y^2) \mathbf{i} + (e^{-y} + x^2) \mathbf{j}$. C consists of the arc of the curve $y = \cos x$ from $(-\frac{\pi}{2}, 0)$ to $(\frac{\pi}{2}, 0)$ and the line segment from $(\frac{\pi}{2}, 0)$ to $(-\frac{\pi}{2}, 0)$.
 - $\mathbf{F}(x,y) = \sqrt{x^2 + 1} \mathbf{i} + \tan^{-1} x \mathbf{j}$. C is the triangle from $(0,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$.
- A particle starts at the point $(-2,0)$, moves along the x -axis to $(2,0)$, and then along the semi-circle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x,y) = x \mathbf{i} + (x^3 + 3xy^2) \mathbf{j}$.
- A circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$. A fixed point P on C traces out an *epicycloid* with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Sketch the epicycloid and use a line integral to find the area it encloses.
- Let D be a region with area A bounded by a simple closed path C in the xy -plane. Use Green's Theorem to prove that the co-ordinates of the *centroid* (\bar{x}, \bar{y}) of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy, \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 dx$$

Now compute the centroid of a triangle with vertices $(0,0)$, $(a,0)$, and (a,b) , where $a, b > 0$.

- Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = (x^2 + y) \mathbf{i} + (3x - y^2) \mathbf{j}$ and C is the positively-oriented boundary curve of a region D that has area 6.
- (Hard) Let $f(x,y) = \frac{\partial Q}{\partial x}$. Suppose $x = x(u,v)$ and $y = y(u,v)$ map a region S one-one onto a region R in such a way that both regions have simple closed boundaries. Finally, assume that all functions have continuous partial derivatives whenever necessary.

Use Green's Theorem to prove the change of co-ordinates formula

$$\iint_R f(x,y) \, dx \, dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv$$

16.5 Curl and Divergence

1. Find the curl and the divergence of each vector field.

(a) $\mathbf{F}(x, y, z) = xy^2z^3 \mathbf{i} + x^3yz^2 \mathbf{j} + x^2y^3z \mathbf{k}$

(b) $\mathbf{F}(x, y, z) = \sin yz \mathbf{i} + \sin zx \mathbf{j} + \sin xy \mathbf{k}$

(c) $\mathbf{F}(x, y, z) = e^{xy} \sin z \mathbf{j} + y \tan^{-1}(x/z) \mathbf{k}$

(d) $\mathbf{F}(x, y, z) = xy^{-1} \mathbf{i} + yz^{-1} \mathbf{j} + zx^{-1} \mathbf{k}$

2. Let f be a scalar field and \mathbf{F} a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar field.

(a) $\text{curl} f$

(d) $\text{curl}(\text{grad} f)$

(g) $\text{div}(\text{grad} f)$

(j) $\text{div}(\text{div} \mathbf{F})$

(b) $\text{grad} f$

(e) $\text{grad} \mathbf{F}$

(h) $\text{grad}(\text{div} f)$

(k) $(\text{grad} f) \times (\text{div} \mathbf{F})$

(c) $\text{div} \mathbf{F}$

(f) $\text{grad}(\text{div} \mathbf{F})$

(i) $\text{curl}(\text{curl} \mathbf{F})$

(l) $\text{div}(\text{curl}(\text{grad} f))$

3. Check that $\nabla \cdot \nabla \times \mathbf{F} = 0$ for any vector field \mathbf{F} with continuous second derivatives.

4. Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F}(x, y, z) = xyz^2 \mathbf{i} + x^2yz^2 \mathbf{j} + x^2y^2z \mathbf{k}$

(b) $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z \mathbf{j} + y \cos z \mathbf{k}$

5. Is there a vector field \mathbf{G} on \mathbb{R}^3 such that $\text{curl} \mathbf{G} = xyz \mathbf{i} - y^2z \mathbf{j} + yz^2 \mathbf{k}$? Explain.

6. Prove the identities.

(a) $\text{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \text{div} \mathbf{F}$

(b) $\text{curl}(f\mathbf{F}) = f \text{curl} \mathbf{F} + \nabla f \times \mathbf{F}$

(c) $\text{div}(\nabla f \times \nabla g) = 0$

7. Let $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ and $r = |\mathbf{r}|$.

(a) Establish the following.

i. Show that $\nabla \cdot \mathbf{r} = 3$.

ii. $\nabla \cdot (r\mathbf{r}) = 4r$

iii. $\nabla^2 r^3 = 12r$

(You can compute these directly, but you may find it helpful first to show that $\nabla f(r) = \frac{f'(r)}{r} \mathbf{r}$)

(b) If $\mathbf{F} = r^{-p} \mathbf{r}$, find $\text{div} \mathbf{F}$. Is there a value of p for which $\text{div} \mathbf{F} = 0$?