Math 2E Suggested Written Questions 16.7, 16.8

16.7 Surface Integrals

1. Evaluate the surface integral.

   (a) $\iint_S xyz \, dS$ where S is the cone with parametric equations

   $$x = u\cos v, \quad y = u\sin v, \quad z = u, \quad 0 \le u \le 1, \quad 0 \le v \le \frac{\pi}{2}$$

   (b) $\iint_S xz \, dS$ where S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant.

   (c) $\iint_S y \, dS$ where S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2}), \ 0 \le x, y \le 1$.

   (d) $\iint_S xz \, dS$ where S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$.
   (*You'll have to consider three surfaces separately!*)

2. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces use the positive (outward) orientation.

   (a) $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$, S is the helicoid with vector equation

   $$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}, \quad 0 \le u \le 1, \quad 0 \le v \le \pi$$

   with upward orientation.

   (b) $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 25, \ y \ge 0$, oriented in the direction of the positive y-axis.

   (c) $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + 4x^2\,\mathbf{j} + yz\,\mathbf{k}$, S is the surface $z = xe^y, \ 0 \le x, y \le 1$, with upward orientation.

   (d) $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$, S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

3. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$ for $1 \le z \le 4$, if its density function is $\rho(x, y, z) = 10 - z$.

4. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$ where x, y, z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$ where $z \ge 0$.

5. The temperature at a point in a ball (radius R) with thermal conductivity k is inversely proportional to the distance $r \le R$ from the center of the ball

   $$T(r) = \frac{c}{r}$$

   Find the rate of heat flow across a sphere S of radius $a \le R$ whose center is that of the ball.

16.8 Stokes Theorem

1. Use Stokes Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

   (a) $\mathbf{F}(x, y, z) = 2y \cos z\, \mathbf{i} + e^x \sin z\, \mathbf{j} + xe^y\, \mathbf{k}$ where S is the hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$, oriented upward.

   (b) $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\, \mathbf{i} + x^2y\, \mathbf{j} + x^2z^2\, \mathbf{k}$ where S is the cone $x = \sqrt{y^2 + z^2}$ with $0 \leq x \leq 2$, oriented in the direction of the positive x-axis.

   (c) $\mathbf{F}(x, y, z) = e^{xy}\, \mathbf{i} + e^{xz}\, \mathbf{j} + x^2z\, \mathbf{k}$ where S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ to the right of the xz-plane, oriented in the direction of the positive y-axis.

2. Use Stokes Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In both cases C is oriented counterclockwise when viewed from above.

   (a) $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\, \mathbf{j} + (xy - \sqrt{z})\, \mathbf{k}$ where C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant.

   (b) $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + 2z\, \mathbf{j} + 3y\, \mathbf{k}$ where C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$.

3. (a) Use Stokes Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x^2y\, \mathbf{i} + \frac{1}{3}x^3\, \mathbf{j} + xy\, \mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise when viewed from above.

   (b) Sketch both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

   (c) Find parametric equations for C and use them to graph C.

4. Verify Stokes Theorem for the vector field $\mathbf{F}(x, y, z) = -2yz\, \mathbf{i} + y\, \mathbf{j} + 3x\, \mathbf{k}$ and the surface S, the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward.

5. Let C be a simple closed smooth curve in the plane $x + y + z = 1$. Show that the line integral

   $$\int_C z\, dx - 2x\, dy + 3y\, dz$$

   depends only on the area of the region enclosed by C and not otherwise on its shape or location in the plane.

6. Evaluate

   $$\int_C (y + \sin x)\, dx + (z^2 + \cos y)\, dy + x^3\, dz$$

   where C is the curve $\mathbf{r}(t) = \sin t\, \mathbf{i} + \cos t\, \mathbf{j} + \sin 2t\, \mathbf{k}$ for $0 \leq t \leq 2\pi$.

   (*Hint: Observe that C lies on the surface* $z = 2xy$)

7. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

   (*Intuitively this is obvious but a rigorous proof requires some limits…*)