Math 2E Suggested Written Questions 16.9

16.9 Divergence Theorem

1. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

   (a) $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ where E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

   (b) $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} - y\,\mathbf{j} + z\,\mathbf{k}$ where E is the solid cylinder $y^2 + z^2 \leq 9$ for $0 \leq x \leq 2$

2. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

   (a) $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$. S is the surface of the box enclosed by the planes $x = 0, a$, $y = 0, b$, and $z = 0, c$, where a, b, c are positive numbers.

   (b) $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$. S is the sphere of radius 2 centered at the origin.

   (c) $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$. S is the surface of the tetrahedron enclosed by the co-ordinate planes and the plane

   $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

   where a, b, c are positive numbers.

   (d) $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} - x^3z^2\,\mathbf{j} + 4xy^2z\,\mathbf{k}$. S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$.

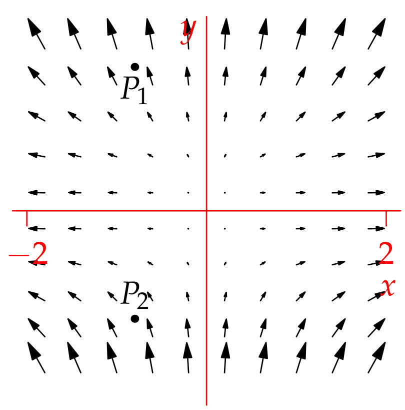
   (e) $\mathbf{F} = |\mathbf{r}|^2\,\mathbf{r} = r^2\mathbf{r}$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is the sphere of radius R centered at the origin

3. Let $\mathbf{F}(x, y, z) = z\tan^{-1}(y^2)\,\mathbf{i} + z^3\ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

4. Consider the vector field in the picture.

   (a) Are the points P_1 and P_2 sources or sinks? Give an explanation based solely on the figure.

   (b) Given that $\mathbf{F} = x\,\mathbf{i} + y^2\,\mathbf{j}$, use the definition of divergence to verify your answer to part (a).



5. Use the Divergence Theorem to evaluate

$$\iint_S 2x + 2y + z^2 \, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$

6. Supposing that S and E satisfy the conditions of the Divergence Theorem and that f has continuous partial derivatives, prove that

$$\iint_S D_{\mathbf{n}} f \, \mathrm{d}S = \iiint_E \nabla^2 f \, \mathrm{d}V$$

(Here $D_{\mathbf{n}} f = \nabla f \cdot \mathbf{n}$ is the *directional derivative* of f)

7. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f \mathbf{n} \, \mathrm{d}S = \iiint_E \nabla f \, \mathrm{d}V$$

Each integral is computed by integrating separately each component function of the vectors.

(*Hint: start by applying the Divergence Theorem to* $\mathbf{F} = f\mathbf{i}\ldots$)

8. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. Set up a co-ordinate system so that the xy-plane is the surface of the liquid and positive values of z measure depth downward into the liquid.

The pressure at depth z is modeled by $p = \rho g z$, where g is the acceleration due to gravity. The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n} \, \mathrm{d}S$$

where $\mathbf{n}$ is the outward-pointing unit normal. Use the result of Exercise 7 to show that $\mathbf{F} = -W\mathbf{k}$ where W is the weight of the liquid displaced by the solid.

This is *Archimedes' Principle:* the buoyancy force equals the weight of the displaced liquid.