Math 2E Multi-Variable Calculus Pre-midterm Questions

16.2 Line Integrals

4 Evaluate the line integral $\int_C x \sin y \, ds$ where C is the line segment from (0,3) to (4,6).

8 Evaluate the line integral $\int_C x^2 \, dx + y^2 \, dy$ where C consists of the $\frac{1}{4}$-arc of the circle $x^2 + y^2 = 4$ from (2,0) to (0,2) followed by the line segment from (0,2) to (4,3).

10 Evaluate the line integral $\int_C xyz^2 \, ds$ where C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

14 Evaluate the line integral $\int_C y \, dx + z \, dy + x \, dz$ where C is parameterized by

$$x = \sqrt{t}, \quad y = t, \quad z = t^2, \quad \text{for } 1 \le t \le 4.$$

20 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y - z)\mathbf{j} + z^2\mathbf{k}$, and C is parameterized by $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k}$ for $0 \le t \le 1$.

22 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xy\mathbf{k}$, and C is parameterized by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ for $0 \le t \le \pi$.

28 Use a graph of the vector field $\mathbf{F}(x, y) = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix}$ and the curve C, the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$, to guess whether the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative or zero. Then evaluate the line integral.

32 (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented counter-clockwise.

(b) Sketch a graph of the force field and the circle. Use it to explain your answer to part (a).

40 Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + ye^x\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from (1,0) to (2,1).

42 The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is

$$\mathbf{F}(\mathbf{r}) = \frac{K\mathbf{r}}{|\mathbf{r}|^3}$$

where K is a constant. Find the work done as the particle moves along a straight line from (2,0,0) to (2,1,5).

44 An object with mass m moves with position function $\mathbf{r}(t) = a \sin t\mathbf{i} + b \cos t\mathbf{j} + ct\mathbf{k}$ for $0 \le t \le \frac{\pi}{2}$. Find the work done (by gravity) on the object during this time period.

48 The base of a circular fence of radius 10 m is given by $x = 10 \cos t$, $y = 10 \sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m^2. Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

50 If C is a smooth curve given by a vector function $\mathbf{r}(t)$ for $a \le t \le b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \frac{1}{2} \left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2 \right].$$

16.3 The Fundamental Theorem for Line Integrals

4–8 Determine whether $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

4 $\mathbf{F}(x, y) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$

6 $\mathbf{F}(x, y) = (3x^2 - 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$

8 $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j},\ y > 0$

12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along the given curve C

12 $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$. C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.

16 $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$. C: $x = \sqrt{t},\ y = t + 1,\ z = t^2,\ 0 \le t \le 1$.

18 $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z\,\mathbf{k}$. C: $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le \frac{\pi}{2}$.

20 Show that the integral $\int_C \sin y\,\mathrm{d}x + (x \cos y - \sin y)\,\mathrm{d}y$ is independent of path and evaluate it if C is any path from $(2, 0)$ to $(1, \pi)$.

22 Suppose an experiment determines that the amount of work required for a force field $\mathbf{F}$ to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by $\mathbf{F}$ in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about $\mathbf{F}$? Why?

24 Find the work done by the force field $\mathbf{F} = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ in moving a particle from $(0, 1)$ to $(2, 0)$.

30 Show that the line integral $\int_C y\,\mathrm{d}x + x\,\mathrm{d}y + xyz\,\mathrm{d}z$ is not independent of path.

32, 34 Determine whether the given set is (a) open, (b) connected, and (c) simply-connected.

32 $\{(x, y) : 1 < |x| < 2\}$

34 $\{(x, y) : (x, y) \ne (2, 3)\}$

36 (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3} = \frac{c}{r^3}\mathbf{r}$$

for some constant c, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Find the work done in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}$. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^{11} m from the sun) to perihelion (at a minimum distance of 1.47×10^{11} m). Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$.

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \frac{\varepsilon qQ}{r^3}\mathbf{r}$. Suppose that an electron with a charge of $Q = -1.6 \times 10^{-19}$ C is located at the origin. A proton with charge $q = 1.6 \times 10^{-19}$ C is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. Use the value $\varepsilon = 8.985 \times 10^9\ \mathrm{Nm^2/C^2}$.

16.4 Green's Theorem

2, 4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem

2 $\oint_C xy\,\mathrm{d}x + x^2\,\mathrm{d}y$. C is the rectangle with vertices (0,0), (3,0), (3,1), and (0,1).

4 $\oint_C x^2y^2\,\mathrm{d}x + xy\,\mathrm{d}y$. C consists of the arc of the parabola $y = x^2$ from (0,0) to (1,1) and the line segments from (1,1) to (0,1) and from (0,1) to (0,0).

6, 10 Use Green's Theorem to evaluate the line integral along the positively oriented curve given

6 $\oint_C \cos y\,\mathrm{d}x + x^2 \sin y\,\mathrm{d}y$. C is the rectangle with vertices (0,0), (5,0), (5,2), and (0,2).

10 $\oint_C (1 - y^3)\,\mathrm{d}x + (x^3 + e^{y^2})\,\mathrm{d}y$. C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

12, 14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. Check the orientation of the curve before applying the theorem.

12 $\mathbf{F}(x, y) = (e^{-x} + y^2)\mathbf{i} + (e^{-y} + x^2)\mathbf{j}$. C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$.

14 $\mathbf{F}(x, y) = \sqrt{x^2 + 1}\,\mathbf{i} + \tan^{-1} x\,\mathbf{j}$. C is the triangle from (0,0) to (1,1) to (0,1) to (0,0).

18 A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = x\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$.

20–28 Not covered on Midterm. Will be on Final Exam.

20 If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5\cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use a line integral to calculate the area it encloses.

22 Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the co-ordinates of the centroid $(\overline{x}, \overline{y})$ of D are

$$\overline{x} = \frac{1}{2A} \oint_C x^2\,\mathrm{d}y, \quad \overline{y} = -\frac{1}{2A} \oint_C y^2\,\mathrm{d}x$$

where A is the area of D.

24 Use question 22 to find the centroid of a triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a, b > 0$.

28 Calculate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (3x - y^2)\mathbf{j}$ and C is the positively oriented boundary curve of a region D that has area 6.