Math 2E Multi-Variable Calculus Homework Questions 2

15.8 Triple Integrals in Cylindrical Co-ordinates

4 Convert the points $(2\sqrt{3}, 2, -1)$ and $(4, -3, 2)$ from rectangular to cylindrical co-ordinates.

8 Identify the surface with equation $2r^2 + z^2 = 1$.

18 Evaluate $\iiint_E z\,\mathrm{d}V$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

22 Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

28 Find the mass of a ball B given by $x^2 + y^2 + z^2 \le a^2$, if the density at any point is proportional to its distance from the z-axis.

30 Evaluate the following integral by changing to cylindrical co-ordinates:

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2 + y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

15.9 Triple Integrals in Spherical Polar Co-ordinates

8 Identify the surface whose equation is given in spherical co-ordinates by

$$\rho^2(\sin^2\phi \sin^2\theta + \cos^2\phi) = 9$$

18 Sketch the solid whose volume is given by the following integral, and evaluate it:

$$\int_{0}^{2\pi} \int_{\pi/2}^{\pi} \int_{1}^{2} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

22 Use spherical co-ordinates to evaluate

$$\iiint_H 9 - x^2 - y^2\,\mathrm{d}V$$

where H is the solid hemisphere $x^2 + y^2 + z^2 \le 9$, $z \ge 0$.

26 Use spherical co-ordinates to evaluate

$$\iiint_E xyz\,\mathrm{d}V$$

where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$.

30 Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

40 Evaluate the following integral by changing to spherical co-ordiantes

$$\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2 z + y^2 z + z^3)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

15.10 Change of Variables in Multiple Integrals

4 Find the Jacobian of the transformation

$$x = e^{s+t}, \quad y = e^{s-t}.$$

6 Find the Jacobian of the transformation

$$x = v + w^2, \quad y = w + u^2, \quad z = u + v^2.$$

14 Let R be bounded by the hyperbolas $y = 1/x$, $y = 4/x$ and the lines $y = x$ and $y = 4x$ in the first quadrant. Find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the u- and v-axes.

16 Let R be the parallelogram with corners $(-1, 3)$, $(1, -3)$, $(3, -1)$ and $(1, 5)$. Evaluate the integral

$$\iint_R 4x + 8y \, \mathrm{d}A$$

by using the change of variables $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$.

20 Let R be the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$ and $xy^2 = 2$. Evaluate the integral

$$\iint_R y^2 \, \mathrm{d}A$$

by using the change of variables $u = xy$, $v = xy^2$.

24 Make a suitable change of variables to evaluate the integral $\iint_R (x + y) e^{x^2 - y^2} \, \mathrm{d}A$, where R is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$ and $x + y = 3$.

16 Vector Calculus

16.1 Vector Fields

22 Find the gradient field of the function $f(x, y) = \tan(3x - 4y)$.

24 Find the gradient field of the function $f(x, y, z) = x \ln(y - 2z)$.

26 Find the gradient field ∇f of $f(x, y) = \sqrt{x^2 + y^2}$ and sketch it.

34 At time $t = 1$ a particle is located at position (1,3). If it moves in a velocity field

$$\mathbf{F}(x, y) = \begin{pmatrix} xy - 2 \\ y^2 - 10 \end{pmatrix}$$

find its approximate location at time $t = 1.05$.

35 (Non-examinable: for those who have taken Math 3D) The *flow lines* (or *streamlines*) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

(a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If the parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions must satisfy the differential equations $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = -y$. Solve the differential equations to find an equation of the flow line that passes through the point (1,1).

36 (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ and then sketch some flow lines. What shape do these lines appear to have?

(b) If the parametric equations of a flow line are $x = x(t)$, $y = y(t)$, what differential equations must these functions satisfy? Deduce that $\frac{dy}{dx} = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.