Math 2E Multi-Variable Calculus Homework Questions 4

16.7 Surface Integrals

6–18 Evaluate the surface integral

6 $\iint_S xyz\,\mathrm{d}S$ where S is the cone with parametric equations

$$x = u\cos v, \quad y = u\sin v, \quad z = u, \quad 0 \le u \le 1, \quad 0 \le v \le \pi/2$$

10 $\iint_S xz\,\mathrm{d}S$ where S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant.

12 $\iint_S y\,\mathrm{d}S$ where S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2}),\ 0 \le x, y \le 1$.

18 $\iint_S xz\,\mathrm{d}S$ where S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$.

22–32 Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$ for the given vector field $\mathbf{F}$ and oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces use the positive (outward) orientation

22 $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$, S is the helicoid with vector equation

$$\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le 1, \quad 0 \le v \le \pi$$

with upward orientation.

26 $\mathbf{F}(x, y, z) = xz\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 25,\ y \ge 0$, oriented in the direction of the positive y-axis.

28 $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4x^2\mathbf{j} + yz\mathbf{k}$, S is the surface $z = xe^y,\ 0 \le x, y \le 1$, with upward orientation.

32 $\mathbf{F}(x, y, z) = y\mathbf{i} + (z - y)\mathbf{j} + x\mathbf{k}$, S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

40 Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$ for $1 \le z \le 4$, if its density function is $\rho(x, y, z) = 10 - z$.

44 Seawater has density $1025\ \mathrm{kg/m^3}$ and flows in a velocity field $\mathbf{v} = y\mathbf{i} + x\mathbf{j}$ where x, y, z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \ge 0$.

46 The temperature at a point in a ball with conductivity k is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a a sphere S of radius a with center at the center of the ball.

16.8 Stokes' Theorem

2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$

2 $\mathbf{F}(x,y,z) = 2y\cos z\mathbf{i} + e^x \sin z\mathbf{j} + xe^y\mathbf{k}$ where S is the hemisphere $x^2+y^2+z^2 = 9$ with $z \geq 0$, oriented upward

4 $\mathbf{F}(x,y,z) = \tan^{-1}(x^2yz^2)\mathbf{i} + x^2y\mathbf{j} + x^2z^2\mathbf{k}$ where S is the cone $x = \sqrt{y^2+z^2}$ with $0 \leq x \leq 2$, oriented in the direction of the positive x-axis

6 $\mathbf{F}(x,y,z) = e^{xy}\mathbf{i} + e^{xz}\mathbf{j} + x^2z\mathbf{k}$ where S is the half of the ellipsoid $4x^2+y^2+4z^2 = 4$ to the right of the xz-plane, oriented in the direction of the positive y-axis

8, 10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. In both cases C is oriented counterclockwise when viewed from above.

8 $\mathbf{F}(x,y,z) = \mathbf{i} + (x+yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$ where C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

10 $\mathbf{F}(x,y,z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$ where C is the curve of intersection of the plane $x+z = 5$ and the cylinder $x^2+y^2 = 9$

12 (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F}(x,y,z) = x^2y\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2+y^2 = 1$ oriented counterclockwise when viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

14 Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x,y,z) = -2yz\mathbf{i} + y\mathbf{j} + 3x\mathbf{k}$ and the surface S, the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward.

16 Let C be a simple closed smooth curve that lies in the plane $x+y+z = 1$. Show that the line integral

$$\int_C z\,\mathrm{d}x - 2x\,\mathrm{d}y + 3y\,\mathrm{d}z$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

18 Evaluate

$$\int_C (y+\sin x)\,\mathrm{d}x + (z^2+\cos y)\,\mathrm{d}y + x^3\,\mathrm{d}z$$

where C is the curve $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \sin 2t\mathbf{k}$ for $0 \leq t \leq 2\pi$. (*Hint:* Observe that C lies on the surface $z = 2xy$.)

19 If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0$ (*Intuitively this is obvious but a full proof requires some upper-division level analysis - beware...*)

16.9 The Divergence Theorem

2, 4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E

2 $\mathbf{F}(x,y,z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ where E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

4 $\mathbf{F}(x,y,z) = x^2\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ where E is the solid cylinder $y^2 + z^2 \leq 9$ for $0 \leq x \leq 2$

6–14 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

6 $\mathbf{F}(x,y,z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$. S is the surface of the box enclosed by the planes $x = 0, a$, $y = 0, b$, and $z = 0, c$, where a, b, c are positive numbers.

8 $\mathbf{F}(x,y,z) = (x^3 + y^3)\mathbf{i} + (y^3 + z^3)\mathbf{j} + (z^3 + x^3)\mathbf{k}$. S is the sphere of radius 2 centered at the origin.

10 $\mathbf{F}(x,y,z) = z\mathbf{i} + y\mathbf{j} + zx\mathbf{k}$. S is the surface of the tetrahedron enclosed by the co-ordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, c are positive numbers.

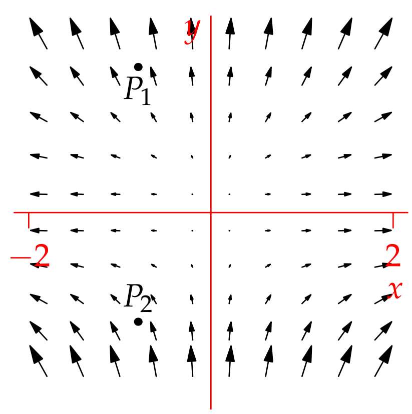
12 $\mathbf{F}(x,y,z) = x^4\mathbf{i} - x^3z^2\mathbf{j} + 4xy^2z\mathbf{k}$. S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$.

14 $\mathbf{F} = |\mathbf{r}|^2\,\mathbf{r} = r^2\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and S is the sphere of radius R centered at the origin

18 Let $\mathbf{F}(x,y,z) = z\tan^{-1}(y^2)\mathbf{i} + z^3\ln(x^2+1)\mathbf{j} + z\mathbf{k}$. Find the flux of $\mathbf{F}$ across the the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

20 (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the picture? Give an explanation based solely on the figure.

(b) Given that $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j}$, use the definition of divergence to verify your answer to part (a).



24 Use the Divergence Theorem to evaluate

$$\iint_S 2x + 2y + z^2\,\mathrm{d}S$$

where S is the sphere $x^2 + y^2 + z^2 = 1$

28 Supposing that S, E satisfy the conditions of the Divergence Theorem and that f has continuous partial derivatives, prove that

$$\iint_S D_{\mathbf{n}} f \, \mathrm{d}S = \iiint_E \nabla^2 f \, \mathrm{d}V$$

(Here $D_{\mathbf{n}} f = \nabla f \cdot \mathbf{n}$ is the *directional derivative* of f)

31 Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n} \, \mathrm{d}S = \iiint_E \nabla f \, \mathrm{d}V$$

Here each integral is computed as by integrating each component function of the vectors. (Hint: start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{i} \ldots$).

32 A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. Set up a co-ordinate system so that the xy-plane is the surface of the liquid and positive values of z measure depth downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity. The total bouyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n} \, \mathrm{d}S$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 31 to show that $\mathbf{F} = -W\mathbf{k}$ where W is the weight of the liquid displaced by the solid. This is *Archimedes' Principle:* the bouyancy force equals the weight of the displaced liquid.