Math 2E: Vector Calculus

Final Exam (v1) (44415)

December 17th 2014
1.30–3.30pm

Name:

Student Id#:

Total marks = 100 (per question in brackets)
No calculators or other electronic devices
Unless otherwise stated, include all your working for full credit
No leaving the exam in the last 10 minutes (be courteous to those finishing off)
Try all parts of every question, even if you can't do the first part!

Question	Marks
1	/11
2	/17
3	/15
4	/20
5	/20
6	/17
Total	/100

1. The surface S is the part of the graph of $z = x^2 + 2y$ lying above the triangle (in the xy-plane) with vertices $(0,0)$, $(1,0)$ and $(1,2)$.

   (a) In Cartesian co-ordinates, show that (6)

   $$\mathrm{d}S = \sqrt{5 + 4x^2}\,\mathrm{d}x\,\mathrm{d}y.$$

   (b) Prove that the surface area of S is $\dfrac{27 - 5\sqrt{5}}{6}$ units^2. (5)

2. (a) Sketch the surface S defined by $y = 9 - x^2 - z^2$, where $x^2 + z^2 \leq 9$. (4)

(b) Parameterize S using polar co-ordinates (r, θ) *in the xz-plane.* (3)

(c) Compute $\mathrm{d}\mathbf{S}$, supposing that S is oriented in the direction of the *positive y-axis*. (4)

(d) Now compute the integral $\displaystyle\iint_S \begin{pmatrix} z \\ -2 \\ -x \end{pmatrix} \cdot \mathrm{d}\mathbf{S}$ (6)

3. Consider the vector field

$$\mathbf{F}(x, y, z) = e^y \mathbf{i} + (xe^y + e^z)\mathbf{j} + ye^z \mathbf{k}$$

(a) Show that $\mathbf{F}$ is conservative by finding a potential function (4)

(b) Calculate the work done by $\mathbf{F}$ in moving a particle along the straight line segment C from $(0, 2, 0)$ to $(4, 0, 3)$. (5)

(c) Use your answer to part (a) to help evaluate (6)

$$\int_C 2e^y \,\mathrm{d}x + (xe^y + e^z)\,\mathrm{d}y + ye^z \,\mathrm{d}z$$

along the *same* curve C. (*Think about breaking this into two integrals*)

4. (a) State the Divergence Theorem for a vector field **F** on a volume E. Remember to state all the relevant hypotheses about **F** and E. (6)

(b) Let E be the region between the spheres of radius 1 and 2 centered at the origin, so that the boundary surface of E consists of the two spheres. In the context of the divergence theorem, in what directions are these spheres oriented? (4)

(c) Compute the net outward flux of the vector field $\mathbf{F} = xz^2\mathbf{i} + yx^2\mathbf{j} + zy^2\mathbf{k}$ from the region E described in part (b). (10)

5. (a) Let **F** be a vector field with continuous partial derivatives on $\mathbb{R}^3$. Suppose that S is a non-self-intersecting surface with boundary curve C. State Stokes' theorem for the vector field **F** and surface S taking care to explain how the orientations of S and C are related. (5)

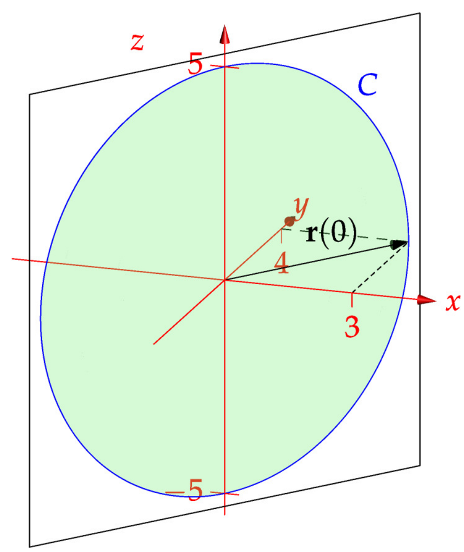
(b) The circle C parameterized by

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 5\sin t\,\mathbf{k}$$

for $0 \leq t \leq 2\pi$ is drawn. Find the equation of the *plane* in which C lies.

Let S be the shaded part of the plane *inside* C. Indicate the orientation of C and the induced orientation of S on the picture.

(*All planes have equation $ax + by + cz = d$ for some constants $a, b, c, d\ldots$*) (4)



(c) Let S and C be as in part (b). Explain why

$$\mathbf{R}(r,\theta) = 3r\cos\theta\mathbf{i} + 4r\cos\theta\mathbf{j} + 5r\sin\theta\mathbf{k}$$

parameterizes S. Compute $\mathrm{d}\mathbf{S}$ and prove that (r,θ) are oriented co-ordinates for the induced orientation on S. (5)

(d) Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}$, where $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y\mathbf{k}$ where C is the circle in part (b). (6)

6. (a) Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on a region $D \subseteq \mathbb{R}^2$. *In terms of P and Q*, what does it mean for $\mathbf{F}$ to be:

(i) Irrotational (4)

(ii) Incompressible (4)

(b) Consider the vector field

$$\mathbf{F}(x, y) = \frac{x - 2y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y + 2x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

defined on $\mathbb{R}^2$ without the origin. Show that $\mathbf{F}$ is *not* irrotational. (5)

(c) The vector field in part (b) is sketched below. A rubber duck is placed at the point A. The duck drifts with the vector field along the dashed path, reaching B after one second and C after 2 seconds.
Draw the duck when it reaches points B and C, including your best guess as to the *direction* in which the duck is pointing. Explain why you have drawn the duck as you have, making sure to reference your answer to part (b). (4)