Linear Algebra Chapter Tests B

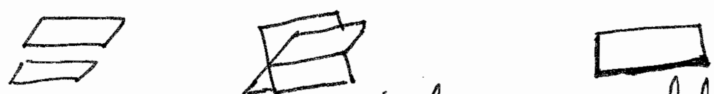
Ch 1

1/ $\left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & -2 & 7 & 7 & 1 \end{array}\right) \underset{-2R_1}{\overset{+R_1}{\leadsto}} \left(\begin{array}{cccc|c} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right) \underset{-R_2}{\overset{-3R_2}{\leadsto}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -7 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 0 \\ -3 \\ 1 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

2a/ Plane in 3-dimensional space.

b/ 0 or ∞ solutions:



2 // planes $\Rightarrow$ 0 sol$^{\underline{n}}$s — planes intersect in line ∞ sol$^{\underline{n}}$s — repeated plane ∞ solutions.

c/ ∞ sol$^{\underline{n}}$s: homogeneous systems always have solutions.

3a/ ∞ solutions eg: $A(\underline{x}_1 + \lambda(\underline{x}_2 - \underline{x}_1)) = A\underline{x}_1 + \lambda A\underline{x}_2 - \lambda A\underline{x}_1 = \underline{b}$ for all $\lambda \in \mathbb{R}$.

b/ Yes: $A(\underline{x}_2 - \underline{x}_1) = 0$ so $A\underline{x} = 0$ has a $\neq 0$ sol$^{\underline{n}}$ $\underline{x}_2 - \underline{x}_1 \Rightarrow A$ singular.

4/ a/ $A\underline{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ reads $\alpha + \beta = 3 = \frac{1}{2}$ – clearly a contradiction.

b/ $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ makes $A\underline{x} = \underline{b}$ consistent $\Leftrightarrow b_2 = 2b_1$. 2nd row of A being twice first $\Rightarrow$ 2nd equation must be twice first for consistency.

5/ a/ $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, b/ ignore – column operation question!

6/ Yes: $\underline{A}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \underline{b}$

7/ Yes: $A\underline{x} = 0$ has a $\neq 0$ solution, namely $\underline{x} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

8/ Yes; e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\Rightarrow A\underline{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = B\underline{x}_0$.

9/ No: eg. $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ is not symmetric

10/ Yes: $C^{-1} = F^{-1}E^{-1}$ is a product of elementary matrices since inverse of elem matrices are elementary.

②

11/ $A^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -B & I \end{pmatrix}$

12/ $A = \begin{array}{c|cc} & 5 & 5 \\ \hline 6 & A_{11} & A_{12} \\ 4 & A_{21} & A_{22} \end{array}$ $\quad B = \begin{array}{c|cc} & r & 10-r \\ \hline 5 & B_{11} & B_{12} \\ 5 & B_{21} & B_{22} \end{array}$

$k=5$ no condition on r.

Ch 2

1/ a/ $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = \frac{1}{8} \cdot 4 = \frac{1}{2}$

b/ $\det(B^{-1}A^T) = \frac{\det A}{\det B} = \frac{4}{6} = 2/3$

c/ $\det(EA^2) = \det E (\det A)^2 = -1 \cdot 4^2 = -16$

2/ a/ $\det A = x^3 + 1 - 1 - x - x + x = x^3 - x$

b/ A singular $\Leftrightarrow \det A = 0 \Leftrightarrow x(x^2-1) = 0 \Leftrightarrow x = 0, \pm 1$.

3/ Ignore - LU factorization not in course ...

4/ $\det(A^TA) = (\det A)^2 > 0 \Rightarrow A^TA$ non-singular.

5/ $\det B = \det(S^{-1}AS) = (\det S)^{-1} \det A \det S = \det A$.

6/ $\det C = \det A \det B$ so C singular $\Rightarrow \det A \cdot \det B = 0 \Rightarrow \det A$ or $\det B = 0$
$\Rightarrow A$ or B singular.

7/ $\det(A - \lambda I) = 0 \Leftrightarrow (A - \lambda I)\underline{x} = 0$ for some $\underline{x} \neq 0$ (ie $A - \lambda I$ singular)
$\Leftrightarrow A\underline{x} = \lambda \underline{x}$ for some $\underline{x} \neq 0$.

8/ Note that $A\underline{x} = \underline{x}\underline{y}^T\underline{x} = (\underline{y}^T\underline{x})\underline{x}$ and $A\underline{y} = (\underline{y}^T\underline{y})\underline{x}$. If $\underline{y}^T\underline{x} = 0$ then $A\underline{x} = 0$ so $\det A = 0$.
Otherwise $A\left(\frac{1}{\underline{y}^T\underline{y}}\underline{y} - \frac{1}{\underline{y}^T\underline{x}}\underline{x}\right) = \underline{x} - \underline{x} = 0 \quad \therefore A$ is singular (unless $\underline{y} \parallel \underline{x}$).

Easier: let $(a_1, \dots, a_n)$ be any $\neq 0$ sol^n to $a_1y_1 + a_2y_2 + \dots + a_ny_n = 0$.
Then $A\underline{a} = 0$ so A is singular $\Rightarrow \det A = 0$

9/ $A(\underline{x} - \underline{y}) = 0$ with $\underline{x} - \underline{y} \neq 0 \Rightarrow A$ singular $\Rightarrow \det A = 0$.

10/ $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \pm \operatorname{adj} A$ is a matrix of integers, since A is.

Ch6/

1/a) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & -1 \\ 1 & 2 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2+\lambda) = (1-\lambda)(1+\lambda)\lambda = 0$

$\Rightarrow \lambda = 0, \pm 1.$

b) $\underline{\lambda = 0}$: $A\underline{v} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{pmatrix}\underline{v} = 0 \Leftrightarrow \underline{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = 1}$: $(A-I)\underline{v} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & -3 \end{pmatrix}\underline{v} = 0 \Leftrightarrow \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda = -1}$: $(A+I)\underline{v} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}\underline{v} = 0 \Leftrightarrow \underline{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

c) $A = XDX^{-1}$ where $X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$A^7 = XD^7X^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1^7 & 0 \\ 0 & 0 & (-1)^7 \end{pmatrix}X^{-1} = A.$

2/ One of the remaining e-values is $3-2i$ ($A\underline{v} = \lambda\underline{v} \Rightarrow \overline{A\underline{v}} = \overline{\lambda\underline{v}}$

$\Rightarrow A\bar{\underline{v}} = \bar{\lambda}\bar{\underline{v}}$

$\Rightarrow \bar{\lambda}$ an e-value)

The remaining e-values are either both real, or a complex conjugate pair.

3/ a/ If $\lambda = 0$, then $A\underline{v} = 0$ for some $\underline{v} \neq 0 \Rightarrow A$ singular. Contradiction.

b/ $A\underline{v} = \lambda\underline{v} \Rightarrow \underline{v} = A^{-1}\lambda\underline{v} = \lambda A^{-1}\underline{v} \Rightarrow A^{-1}\underline{v} = \lambda^{-1}\underline{v}$.

4/ $|A-\lambda I| = (a-\lambda)^3 = 0 \Leftrightarrow \lambda = a$.

$(A-aI)\underline{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}\underline{v} = 0 \Leftrightarrow \underline{v} = \alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$.

Only one eigenspace of dimension 2 $\Rightarrow$ A defective.