Math 3A: Homework 1

Submit these questions at the discussion on Thursday 6th October

1. For each of the following systems of equations:
   - Identify the system as square, underdetermined or overdetermined.
   - Write the system in augmented matrix format and find a row echelon form.
   - Identify the lead and free variables.
   - Decide whether the system is consistent and, if so, find all of its solutions.

   (a) \[
   \begin{align*}
   x_1 + 2x_2 &= 4 \\
   -2x_1 - 2x_2 &= 4
   \end{align*}
   \]

   (b) \[
   \begin{align*}
   x_1 + x_2 + x_3 + x_4 &= 1 \\
   2x_2 + x_3 - 2x_4 &= 1 \\
   4x_3 + x_4 &= -2 \\
   x_4 &= -3
   \end{align*}
   \]

   (c) \[
   \begin{align*}
   5x_1 - 2x_2 + x_3 &= 3 \\
   2x_1 + 3x_2 - 4x_3 &= 0
   \end{align*}
   \]

   (d) \[
   \begin{align*}
   3x_1 + 2x_2 &= 8 \\
   x_1 + 5x_2 &= 7 \\
   x_1 - 8x_2 &= -6
   \end{align*}
   \]

   (e) \[
   \begin{align*}
   x_1 - 3x_2 + x_3 &= 1 \\
   2x_1 + x_2 - x_3 &= 2 \\
   x_1 + 4x_2 - 2x_3 &= 1 \\
   5x_1 - 8x_2 + 2x_3 &= 6
   \end{align*}
   \]

   (f) \[
   \begin{align*}
   x_1 + x_2 + x_3 &= 0 \\
   x_1 - x_2 - x_3 &= 0
   \end{align*}
   \]

2. Consider the system with augmented matrix

\[
\begin{pmatrix}
1 & 1 & 3 & 2 \\
0 & 1 & 1 & 1 \\
1 & 3 & a & b
\end{pmatrix}
\]

where \(a\) and \(b\) are constants.

   (a) Find a row echelon form of this system.

   (b) Find conditions on \(a\) and \(b\) such that:

   i. The system has a unique solution.

   ii. The system is inconsistent.

   iii. The system has infinitely many solutions.

3. Consider the system of equations

\[
\begin{align*}
ax - y &= p \\
by - y &= q
\end{align*}
\]

where \(a, b, p, q\) are constants. Geometrically, each equation represents a straight line in the \(x, y\)-plane.
(a) What facts about these lines do the values \(a, b, p, q\) represent?

(b) Find conditions on \(a, b, p, q\) such that:
   
   i. The system has a unique solution.
   
   ii. The system is inconsistent.
   
   iii. The system has infinitely many solutions.

What do these conditions mean geometrically? Hint: you may find it easier to treat \(y = x_1\) and \(x = x_2\ldots\)

4. Consider the vectors
   
   \[
   \mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}
   \]

Does the vector equation \(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}\) have a solution?

5. Consider the vectors
   
   \[
   \mathbf{a} = \begin{pmatrix} 2a \\ a \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}
   \]

where \(a\) is a constant. Does there exist a value of \(a\) such that the vector \(\mathbf{a}\) lies in \(\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}\)? Illustrate your answer by making a sketch of the following:

- \(\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}\)
- All of the points in \(\mathbb{R}^3\) whose position vectors are \(\mathbf{a}\) for some value of \(a\).

6. Suppose that \(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\) are the columns of a \(5 \times 3\) matrix \(A\). Suppose also that \(\mathbf{b}\) is a vector satisfying the equations

   \[
   \mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 \quad \text{and} \quad \mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 + 3\mathbf{a}_3
   \]

Prove that the vector equation \(Ax = \mathbf{b}\) has infinitely many solutions.

7. Find a non-trivial linear combination of the vectors
   
   \[
   \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}
   \]

such that \(x_1\mathbf{v}_1 + \cdots + x_4\mathbf{v}_4 = \mathbf{0}\). That is, at least some of the \(x_i\) must be non-zero.