Math 3A: Homework 1

Submit these questions at the discussion on Thursday 6th October

1. For each of the following systems of equations:

   - Identify the system as square, underdetermined or overdetermined.
   - Write the system in augmented matrix format and find a row echelon form.
   - Identify the lead and free variables.
   - Decide whether the system is consistent and, if so, find all of its solutions.

   (a) $$\begin{cases} x_1 + 2x_2 &= 4 \\ -2x_1 - 2x_2 &= 4 \end{cases}$$

   (b) $$\begin{cases} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_2 + x_3 - 2x_4 &= 1 \\ 4x_3 + x_4 &= -2 \\ x_4 &= -3 \end{cases}$$

   (c) $$\begin{cases} 5x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 - 4x_3 &= 0 \end{cases}$$

   (d) $$\begin{cases} 3x_1 + 2x_2 &= 8 \\ x_1 + 5x_2 &= 7 \\ x_1 - 8x_2 &= -6 \end{cases}$$

   (e) $$\begin{cases} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 2 \\ x_1 + 4x_2 - 2x_3 &= 1 \\ 5x_1 - 8x_2 + 2x_3 &= 6 \end{cases}$$

   (f) $$\begin{cases} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0 \end{cases}$$

2. Consider the system with augmented matrix

   $$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & a & b \end{array}\right)$$

   where a and b are constants.

   (a) Find a row echelon form of this system.

   (b) Find conditions on a and b such that:

      i. The system has a unique solution.

      ii. The system is inconsistent.

      iii. The system has infinitely many solutions.

3. Consider the system of equations

   $$\begin{cases} ax - y = p \\ bx - y = q \end{cases}$$

   where a, b, p, q are constants. Geometrically, each equation represents a straight line in the x, y-plane.

(a) What facts about these lines do the values a, b, p, q represent?

(b) Find conditions on a, b, p, q such that:

   i. The system has a unique solution.

   ii. The system is inconsistent.

   iii. The system has infinitely many solutions.

   What do these conditions mean *geometrically? Hint: you may find it easier to treat* $y = x_1$ *and* $x = x_2 \ldots$

4. Consider the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix}$$

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution?

5. Consider the vectors

$$\mathbf{a} = \begin{pmatrix} 2a \\ a \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

where a is a constant. Does there exist a value of a such that the *vector* $\mathbf{a}$ lies in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? Illustrate your answer by making a sketch of the following:

- $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$
- All of the points in $\mathbb{R}^3$ whose position vectors are $\mathbf{a}$ for some value of a.

6. Suppose that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of a 5×3 matrix A. Suppose also that $\mathbf{b}$ is a vector satisfying the equations

$$\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 \quad \text{and} \quad \mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 + 3\mathbf{a}_3$$

Prove that the vector equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

7. Find a *non-trivial* linear combination of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

such that $x_1\mathbf{v}_1 + \cdots + x_4\mathbf{v}_4 = \mathbf{0}$. That is, *at least some* of the x_i must be non-zero.