Math 3A: Homework 2

Submit these questions at the discussion on Thursday 13th October

1. Write the solutions to the following systems in parametric vector form: that is, find a vector \( p \) and (possibly) vectors \( v_1, \ldots, v_k \) such that

\[
Ax = b \iff x = p + c_1 v_1 + \cdots + c_k v_k
\]

for any constants \( c_1, \ldots, c_k \).

(a) \[
\begin{align*}
x_1 - 3x_2 &= -5 \\
-2x_1 + 6x_2 &= 10
\end{align*}
\]

(b) \[
\begin{align*}
x_1 + 3x_2 + x_3 &= 6 \\
2x_1 + x_2 &= 5 \\
3x_1 + 4x_2 + x_3 &= 11 \\
5x_2 + 2x_3 &= 7
\end{align*}
\]

(c) \[
\begin{align*}
2x_1 + x_2 - x_3 + x_4 &= 4 \\
3x_1 + 9x_2 - 3x_5 &= 9
\end{align*}
\]

2. Is the following collection of vectors in \( \mathbb{R}^4 \) linearly independent? Justify your answer.

\[
\begin{pmatrix} 1 \\ 3 \\ 6 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}
\]

3. Suppose that

\[
v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 6 \end{pmatrix}
\]

Find a vector \( v_4 \in \mathbb{R}^4 \) such that \( \{v_1, v_2, v_3, v_4\} \) is linearly independent. Justify your answer.

4. Find a \( 2 \times 3 \) matrix \( A \) such that \( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \) is a solution to \( Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

5. Suppose that the solution set of a system of linear equations can be described as

\[
x_1 = 2 + 3x_2, \quad x_3 = 1 - 4x_2
\]

where \( x_2 \) is free. Use vectors to describe this as a line in \( \mathbb{R}^3 \).

6. Balance the following chemical reaction using linear algebra:

\[
\text{MnS} + \text{As}_2\text{Cr}_{10}\text{O}_{35} + \text{H}_2\text{SO}_4 \rightarrow \text{HMnO}_4 + \text{AsH}_3 + \text{Cr}_3\text{O}_{12} + \text{H}_2\text{O}
\]

by considering the numbers (or moles) of atoms of Manganese, Sulphur, Arsenic, Chromium, Oxygen and Hydrogen on each side of the reaction.
7. Prove the following: if \( p \) is a solution to the matrix equation \( Ax = b \), then every other solution has the form \( y = p + n \), where \( An = 0 \).

Hint: Suppose that \( w \) is another solution to \( Ax = b \). What can you say about the vector \( w - p \)?

8. Suppose that \( v \) and \( w \) are two non-parallel vectors in \( \mathbb{R}^3 \). Prove that there exists a vector \( z \in \mathbb{R}^3 \) such that \( \{v, w, z\} \) is a linearly independent set.