Math 3A: Homework 2

Submit these questions at the discussion on Thursday 13th October

1. Write the solutions to the following systems in parametric vector form: that is, find a vector $\mathbf{p}$ and (possibly) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that

$$A\mathbf{x} = \mathbf{b} \iff \mathbf{x} = \mathbf{p} + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

for any constants $c_1, \ldots, c_k$.

(a) $$\begin{cases} x_1 - 3x_2 & = -5 \\ -2x_1 + 6x_2 & = 10 \end{cases}$$

(b) $$\begin{cases} x_1 + 3x_2 + x_3 & = 6 \\ 2x_1 + x_2 & = 5 \\ 3x_1 + 4x_2 + x_3 & = 11 \\ 5x_2 + 2x_3 & = 7 \end{cases}$$

(c) $$\begin{cases} 2x_1 + x_2 - x_3 + x_4 & = 4 \\ 3x_1 + 9x_2 - 3x_5 & = 9 \end{cases}$$

2. Is the following collection of vectors in $\mathbb{R}^4$ linearly independent? Justify your answer.

$$\begin{pmatrix} 1 \\ 3 \\ 6 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

3. Suppose that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

Find a vector $\mathbf{v}_4 \in \mathbb{R}^4$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Justify your answer.

4. Find a 2×3 matrix A such that $\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$ is a solution to $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

5. Suppose that the solution set of a system of linear equations can be described as

$$x_1 = 2 + 3x_2, \qquad x_3 = 1 - 4x_2$$

where x_2 is free. Use vectors to describe this as a line in $\mathbb{R}^3$.

6. Balance the following chemical reaction using linear algebra:

$$\mathsf{MnS} + \mathsf{As_2Cr_{10}O_{35}} + \mathsf{H_2SO_4} \longrightarrow \mathsf{HMnO_4} + \mathsf{AsH_3} + \mathsf{CrS_3O_{12}} + \mathsf{H_2O}$$

by considering the numbers (or moles) of atoms of Manganese, Sulphur, Arsenic, Chromium, Oxygen and Hydrogen on each side of the reaction.

7. Prove the following: if $\mathbf{p}$ is a solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, then every other solution has the form $\mathbf{y} = \mathbf{p} + \mathbf{n}$, where $A\mathbf{n} = \mathbf{0}$.
Hint: Suppose that $\mathbf{w}$ *is another solution to* $A\mathbf{x} = \mathbf{b}$. *What can you say about the vector* $\mathbf{w} - \mathbf{p}$*?*

8. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are two *non-parallel* vectors in $\mathbb{R}^3$. Prove that there exists a vector $\mathbf{z} \in \mathbb{R}^3$ such that $\{\mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is a linearly independent set.