Math 3A: Homework 3

Submit these questions at the discussion on Thursday 3rd November

1. Show, using the definition of subspace, that
$$U = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right\}$$
is a subspace of $\mathbb{R}^3$.

2. Find bases for each the column and null spaces of the following matrices. Also state the rank and nullity, and check that the rank–nullity theorem holds.

   (a) $\begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ -1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & 4 & 5 & 6 \end{pmatrix}$

3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors
$$\begin{pmatrix} 1 \\ -1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -6 \\ 8 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -7 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ -8 \\ 9 \\ -5 \end{pmatrix}$$
What is the dimension of the subspace?

4. Suppose that A is a 6×5 matrix whose null space is *not* $\{\mathbf{0}\}$. What can you say about the rank of A?

5. (a) Prove that $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$.

   (b) Find the co-ordinates of the vector $\mathbf{x} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ with respect to the basis $\mathcal{B}$.

   Note: If $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is the standard basis of $\mathbb{R}^2$, then the co-ordinates of $\mathbf{x}$ with respect to $\mathcal{E}$ is $\mathbf{x}$ itself: that is $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$. We want to compute $[\mathbf{x}]_{\mathcal{B}}$.

6. Find the co-ordinates of $\mathbf{x} = \begin{pmatrix} 8 \\ -9 \\ 4 \end{pmatrix}$ with respect to the basis
$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \right\}$$
of Span $\mathcal{B}$.

7. Suppose that U and V are both subspaces of $\mathbb{R}^n$. We define the following sets:

- The *union* $U \cup V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ or } \mathbf{x} \in V\}$ is the set of all vectors in either U, V or both.
- The *sum* $U + V = \{\mathbf{u} + \mathbf{v} \in \mathbb{R}^n : \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$ is the set of all vectors which can be written as a sum of vectors in U and V.
- The *intersection* $U \cap V = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U \text{ and } \mathbf{x} \in V\}$ is the set of all vectors in both U and V.

(a) Let $U = \text{Span}\{\mathbf{e}_1\}$ and $V = \text{Span}\{\mathbf{e}_2\}$ be subspaces of $\mathbb{R}^2$. Describe the union, sum and intersection of the subspaces U and V. Of the three, which are subspaces of $\mathbb{R}^2$ in their own right?

(b) Let $U = \text{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$ and $V = \text{Span}\{\mathbf{e}_1, \mathbf{e}_3\}$ be subspaces of $\mathbb{R}^3$. Repeat part (a) for these subspaces.

(c) Prove that, in general, the sum and intersection of two subspaces of $\mathbb{R}^n$ are themselves subspaces of $\mathbb{R}^n$.
(*Hopefully parts (a) and (b) have convinced you that the union of two subspaces is not generally a subspace!*)