Math 3A: Homework 4

Submit these questions at the discussion on Tuesday 15th November

1. Use whichever method you like to compute the following determinant:

$$\det \begin{pmatrix} 2 & 1 & 4 & -2 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 4 & -2 & 0 & 4 & 0 \\ 0 & -4 & 0 & 8 & 3 \\ 1 & 1 & 3 & 1 & 2 \end{pmatrix}$$

2. Suppose that A, B, E are 3×3 matrices such that $\det A = 4$, $\det B = 6$ and where E is the elementary matrix swapping rows 1 and 2. Compute the following:

   (a) $\det(\frac{1}{2}A)$ (b) $\det(B^{-1}A^T)$ (c) $\det(EA^2)$

3. Consider the matrix

$$A = \begin{pmatrix} 7 & 2 & 3 \\ 3 & -6 & 15 \\ 1 & 1 & -1 \end{pmatrix}.$$

   Deduce the number of solutions to the system $A\mathbf{x} = \mathbf{0}$. Justify your answer.

4. Use Cramer's rule to find the given values in the solutions to the following linear systems $A\mathbf{x} = \mathbf{b}$:

   (a) Find x_1 if $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -7 \\ 2 \\ 3 \end{pmatrix}$

   (b) Find x_2 if $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 32 \\ 7 \\ 13 \end{pmatrix}$

   (c) Find x_1 and x_3 if $\begin{pmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix}$

5. Suppose that

$$A = \begin{pmatrix} x & -1 & 1 \\ -1 & x & -1 \\ 1 & 1 & x \end{pmatrix}$$

   where x is a real number.

   (a) Compute $\det A$ as a function of x.

   (b) For which values of x is A a singular matrix?

(c) (Challenge) Find the inverse of A whenever x is such that A is non-singular.

6. Suppose that A, B, C are square matrices, that $C = AB$, and that C is singular. Prove that A or B is singular.

7. Suppose that A is a square matrix and that $A\mathbf{x} = A\mathbf{y}$ for some vectors $\mathbf{x} \neq \mathbf{y}$. Prove that A is singular.

8. The *adjoint* matrix adj A of an $n \times n$ invertible matrix A satisfies the matrix equation

$$A(\operatorname{adj} A) = (\det A)I$$

(a) Prove that $\det(\operatorname{adj} A) = (\det A)^{n-1}$.

(b) Suppose that B is a 4×4 matrix whose adjoint has determinant -8. What is the determinant of B?