Math 3A: Homework 6

Submit these questions at the discussion on Tuesday 29th November

1. Compute the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$ of the linear map $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

   (a) Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x, x^2, x^3\}$ with $T : \mathbb{P}^2 \to \mathbb{P}^3$ defined by

   $$T(p)(x) = \int_0^x p(t)\,\mathrm{d}t + p'(x)$$

   (b) Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ with $T : \mathbb{R}^3 \to \operatorname{Span} \mathcal{C}$ defined by

   $$T(\mathbf{x}) = \frac{1}{2}(x_1 + x_2) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3}(x_1 - x_2 + x_3) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

2. Let $T = \frac{\mathrm{d}}{\mathrm{d}x}$ be the function $T : V \to V$ where

   $$V = \operatorname{Span}\{e^{2x} \sin 3x, e^{2x} \cos 3x\}$$

   (a) Prove that T is a linear map.

   (b) Compute the matrix of T with respect to the basis

   $$\mathcal{B} = \{e^{2x} \sin 3x, e^{2x} \cos 3x\}$$

   (c) Use your matrix to compute the integral

   $$\int e^{2x}(\sin 3x + 4 \cos 3x)\,\mathrm{d}x$$

   without using integration by parts.

3. Let $\mathcal{B} = \{2 + 3x, 1 - x\}$ be a basis of $\mathbb{P}^2$.

   (a) Find the co-ordinates $[p]_{\mathcal{B}}$ of the vector $p(x) = 4 - 5x$ with respect to the basis $\mathcal{B}$. More generally, find the change of coördinate matrix $[I]_{\mathcal{E}}^{\mathcal{B}}$ from the standard basis to $\mathcal{B}$.

   (b) Compute the matrix $[T]_{\mathcal{B}} = [I]_{\mathcal{E}}^{\mathcal{B}} [T]_{\mathcal{E}} [I]_{\mathcal{B}}^{\mathcal{E}}$ of the differentiation operator $T = \frac{\mathrm{d}}{\mathrm{d}x}$ with respect to the basis $\mathcal{B}$.

   (c) Check explicitly that

   $$[T]_{\mathcal{B}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff \frac{\mathrm{d}}{\mathrm{d}x}(2 + 3x) = a(2 + 3x) + b(1 - x)$$

   and

   $$[T]_{\mathcal{B}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \iff \frac{\mathrm{d}}{\mathrm{d}x}(1 - x) = c(2 + 3x) + d(1 - x)$$

4. Let $\mathbb{P}^2 = \text{Span}\{1, x, x^2\}$ and let $T : \mathbb{P}^2 \to \mathbb{P}^2$ be defined by

$$T(p) = p'' + (1 + x)p'$$

(a) With respect to the standard basis $\mathcal{E} = \{1, x, x^2\}$, compute the matrix of T.

(b) Find the eigenvalues and eigenvectors of $[T]_{\mathcal{E}}$, and hence diagonalize it.

(c) Find a basis $\mathcal{B}$ of $\mathbb{P}^2$ So that $T(p) = \lambda p$ for each $p \in \mathcal{B}$.

(d) Interpret your answer to part (c) in the following context: For which values of λ does the differential equation

$$p''(x) + (1 + x)p'(x) = \lambda p(x)$$

have a solution $p(x) = a + bx + cx^2$ which is a quadratic polynomial? For each such value of λ, find the solution.