Math 3A: Extra Final Review Questions

1. Solve the following system of equations using any method you like.

$$\begin{cases} x_1 \quad\quad\quad -x_3 + x_4 = -4 \\ 3x_1 \quad\quad\quad -x_3 + x_4 = -2 \\ x_1 + 2x_2 + x_3 - x_4 = 4 \\ \quad\quad x_2 + 2x_3 \quad\quad = 3 \end{cases}$$

2. Compute the subspaces $\mathcal{N}(A), \mathcal{N}(A^T), \text{col}(A), \text{col}(A^T)$ for the following matrix, and check that the fundamental subspace theorem holds. Make a couple of skectches which illustrate the four subspaces and their relationship to one another.

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 3 & 6 \end{pmatrix}$$

3. Suppose that A is non-singular. Prove that $A^T A$ is non-singular.

4. Prove that A is diagonalizable if and only if A^T is.

5. Let A be the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -5 & 7 & 2 \\ 4 & -2 & 2 \end{pmatrix}.$$

   (a) Calculate the eigenvalues and eigenvectors of the matrix A.
   (b) Find a diagonal matrix D and an invertible matrix X such that $A = XDX^{-1}$.

6. Let X be the matrix

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

   (a) Use row operations on the augmented matrix $(X|I)$ to show that the inverse of X is

$$X^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

   (b) The matrix A has the following eigenvalues and eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ with eigenvalue } \lambda_1 = -2,$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ with eigenvalue } \lambda_2 = 4.$$

   Calculate A.

7. Apply the Gram–Schmidt orthogonalization process to compute an orthonormal basis of

$$U = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Hence or otherwise, compute the orthogonal projection of the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ onto U.

8. A square matrix is said to be *orthogonal* if $A^T A = I$ where I is the identity matrix. Prove that the determinant of an orthogonal matrix must be ± 1.

9. The vector space $\mathbb{P}^2 = \text{Span}\{1, x, x^2\}$ together with the function

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x$$

is an *inner product space.*

(a) Find all polynomials $p(x) = a + bx + cx^2$ which are orthogonal to the polynomial $q(x) = 1 - x$: that is for which $\langle p, q \rangle = 0$.

(b) The Gram–Schmidt process applies perfectly well in this inner product space. Apply it to the basis $\{1, x, x^2\}$ to obtain an orthogonal basis of $\mathbb{P}^2$.