Math 3A: Extra Midterm Review Questions

1. A 3×3 system of linear equations represents three planes in three dimensions. The are *eight* general arrangements of these three planes: one resulting in a unique solution, four resulting in no solutions, and three with infinite solutions. For example, the three planes could be distinct and parallel, arranged like three sheets of paper: there is no point common to all three planes and so the system has no solutions.

   What are the other seven arrangements of the three planes, and draw a sketch of each!

2. Let A, B be $n \times n$ matrices. Is it true that

   $$(A - 2B)^2 = A^2 - 4AB + 4B^2?$$

   If not, what should the right hand side be?

3. Consider the following system of equations

   $$\begin{cases} x_1 - 2x_2 + x_3 - 2x_4 = 4, \\ 2x_1 - 4x_2 \phantom{{}+x_3} - 6x_4 = 2, \\ 2x_1 - 4x_2 - x_3 - 7x_4 = -1, \\ 3x_1 - 6x_2 - x_3 - 10x_4 = 0. \end{cases}$$

   (a) Find a Row Echelon form of the augmented matrix of the system.

   (b) List the lead and free variables.

   (c) Write down all the solutions to the system.

4. Let A be the matrix

   $$A := \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}.$$

   Find the inverse of A by using the method of row operations applied to the augmented matrix $(A|I)$.

5. Using any method you like, solve the system $\begin{cases} 3x + 4y + z = 2, \\ -x + y - 2z = 0, \\ 3x + 3y + z = 1. \end{cases}$

6. Consider a system of the form $\begin{cases} ax + by = 0, \\ cx + dy = 0, \end{cases}$ where a, b, c, d are constant scalars.

   (a) What does it mean for a linear system to be consistent?

   (b) Using your definition in part (a), show that the system above is consistent, regardless of the choices of a, b, c, d.

   (c) By considering what each equation represents geometrically, give a *geometric* reason why the system is consistent.

7. Let $A = \begin{pmatrix} a & d & 0 \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix}$ where a, b, c, d, e are constant scalars.

(a) Find a condition on the scalars a, b, c, d, e that is equivalent to A being non-singular.

(b) Using any method you like find the inverse of $\begin{pmatrix} 1 & p & 0 \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}$, where p, q are scalars.

(c) Suppose that the entries of A satisfy the non-singularity condition you found in part (a). By using row operations, or otherwise, find the inverse of A.

Hint: $A = \begin{pmatrix} 1 & \frac{d}{b} & 0 \\ 0 & 1 & \frac{e}{c} \\ 0 & 0 & 1 \end{pmatrix} E_{1,a}E_{2,b}E_{3,c}$ *where* $E_{1,a} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_{2,b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *etc. Thus* $A^{-1} = E_{3,c}^{-1}E_{2,b}^{-1}\cdots$