A Second-Order Existence/Uniqueness Problem (2.1 Second-order Linear Equations)

Consider the equation $x^2y'' - 2xy' + (2 + x^2)y = 0$.

(a) Given that $y_1(x) = x \cos x$ is a solution, guess (and confirm) a second independent solution and thus give the general solution.

(b) Solve the equation with the initial conditions $y(\pi) = 0$ and $y'(\pi) = -2\pi$.

(c) What happens if the initial conditions are $y(0) = 0$ and $y'(0) = 1$?

(d) What about if we just have the initial condition $y(0) = 1$?

Solution

(a) We guess $y_2(x) = x \sin x$, since sines and cosines usually come together! Now check:

$$y_2'(x) = \sin x + x \cos x, \qquad y_2''(x) = 2 \cos x - x \sin x$$

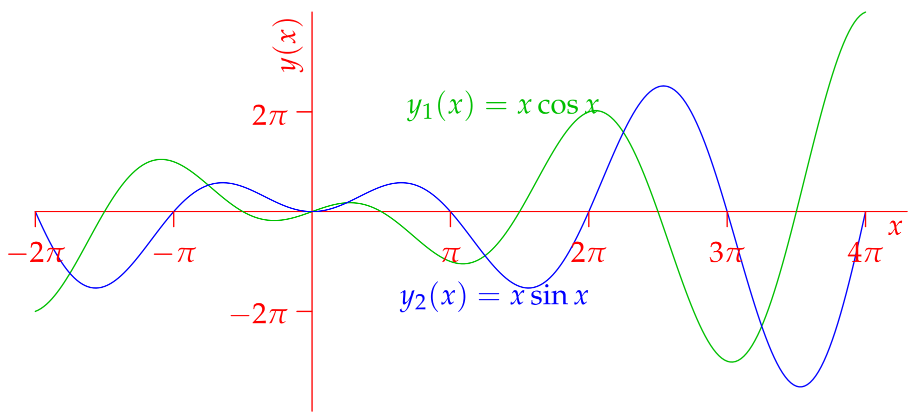
Therefore

$$\therefore x^2 y_2'' - 2xy_2' + (2 + x^2)y_2 = x^2(2 \cos x - x \sin x) - 2x(\sin x + x \cos x) + (2 + x^2)x \sin x = 0$$

Thus $y_2(x)$ is indeed a solution. It is linearly independent to $y_1(x)$, so the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = x(c_1 \cos x + c_2 \sin x).$$

The linearly independent solutions are plotted below:



(b) First calculate

$$y'(x) = (c_1 + c_2 x) \cos x + (c_2 - c_1 x) \sin x$$

Now

$$y(\pi) = -c_1 \pi = 0 \iff c_1 = 0,$$
$$y'(\pi) = -c_1 - c_2 \pi = -2\pi \implies c_2 = 2$$

Therefore the solution is

$$y(x) = 2x \sin x$$

(c) $y(0) = 0$ is not a restriction, since *all* solutions satisfy this. The condition $y'(0) = 1$ forces

$$c_1 = 1$$

whence we have an *infinite family* of solutions

$$y(x) = x(\cos x + c_2 \sin x)$$

Tying this to the existence/uniqueness theorem for second-order linear equations, we may write the original equation as

$$y'' - \frac{2}{x}y' + \left(\frac{2}{x^2} + 1\right)y = 0$$

whence $p(x) = \frac{2}{x}$ and $q(x) = \frac{2}{x^2} + 1$, neither of which are continuous at $x = 0$. The theorem therefore does not apply.

(*In part (b), both p and q are continuous at* $x = \pi$*, hence the unique solution*)

(d) $y(0) = 1$ is impossible given the ODE: if $x = 0$, then we instantly have $2y = 0 \implies y = 0$. The only initial condition with $x = 0$ that can possibly give a solution is $y(0) = 0$. There are thus no solutions. The existence/uniqueness theorem fails for the same reason as in (c).