3.1 Systems of ODES

Example. Two tanks of brine contain $x(t)$ and $y(t)$ lbs of salt at time t, and satisfy the following constraints:

	Tank x	Tank y
Volume (gal)	100	300
Fresh water in (gal/min)	10	0
Brine out (gal/min)	0	10
To other Tank (gal/min)	40	30

We assume that the tanks are well-mixed.

(a) Set up a system of differential equations satisfied by $x(t)$ and $y(t)$.

(b) Solve the system using the method of elimination.

(c) Find and graph the solutions given the initial conditions $x(0) = 10$ lbs and $y(0) = 0$ lbs. When is the mass of salt in tank y maximal?

Solution. (a) $\frac{\mathrm{d}x}{\mathrm{d}t} =$ (salt entering tank x each minute)-(salt leaving), and similarly for y.

Since $\frac{40}{100} = \frac{2}{5}$ of the volume of tank x leaves each minute, and the tanks are well-mixed, it follows that $\frac{2}{5}$ of the salt in x leaves each minute.

Similarly, $\frac{30}{300} = \frac{1}{10}$ of the volume of tank y transfers to tank x each minute, and so a similar proportion of the salt y must also.

Putting this together, with similar equations for y, we have

$$\begin{cases} x'(t) = \frac{-40}{100}x + \frac{30}{300}y = -\frac{2}{5}x + \frac{1}{10}y \\ y'(t) = \frac{40}{100}x - \frac{30}{300}y - \frac{10}{300}y = \frac{2}{5}x - \frac{2}{15}y \end{cases}$$

(b) To solve use the method of elimination, first differentiate the first equation and use the second to substitute for y',:

$$x'' = -\frac{2x'}{5} + \frac{y'}{10} = -\frac{2}{5}x' + \frac{1}{10}\left(\frac{2}{5}x - \frac{2}{15}y\right) = -\frac{2}{5}x' + \frac{1}{25}x - \frac{2}{15}\cdot\frac{1}{10}y.$$

Now use the first equation again to eliminate y:

$$x'' = -\frac{2}{5}x' + \frac{1}{25}x - \frac{2}{15}\cdot\left(x' + \frac{2}{5}x\right) = -\frac{8}{15}x' - \frac{1}{75}x$$

$$\therefore 75x'' + 40x' + x = 0.$$

This may be solved using the characteristic equation method: $\lambda = \frac{-4\pm\sqrt{13}}{15}$:

$$x(t) = e^{-\frac{4}{15}t}\left(ae^{\frac{\sqrt{13}}{15}t} + be^{-\frac{\sqrt{13}}{15}t}\right).$$

where a, b are constants.

Now calculate y using the first equation:

$$y(t) = 10x' + 4x = \tfrac{2}{3}e^{-\frac{4}{15}t}\left(a(2+\sqrt{13})e^{\frac{\sqrt{13}}{15}t} + b(2-\sqrt{13})e^{-\frac{\sqrt{13}}{15}t}\right).$$

(c) Applying the initial conditions $x(0) = 10, y(0) = 0$ we obtain

$$a + b = 10, \qquad \text{and} \qquad a(2+\sqrt{13}) + b(2-\sqrt{13}) = 0,$$

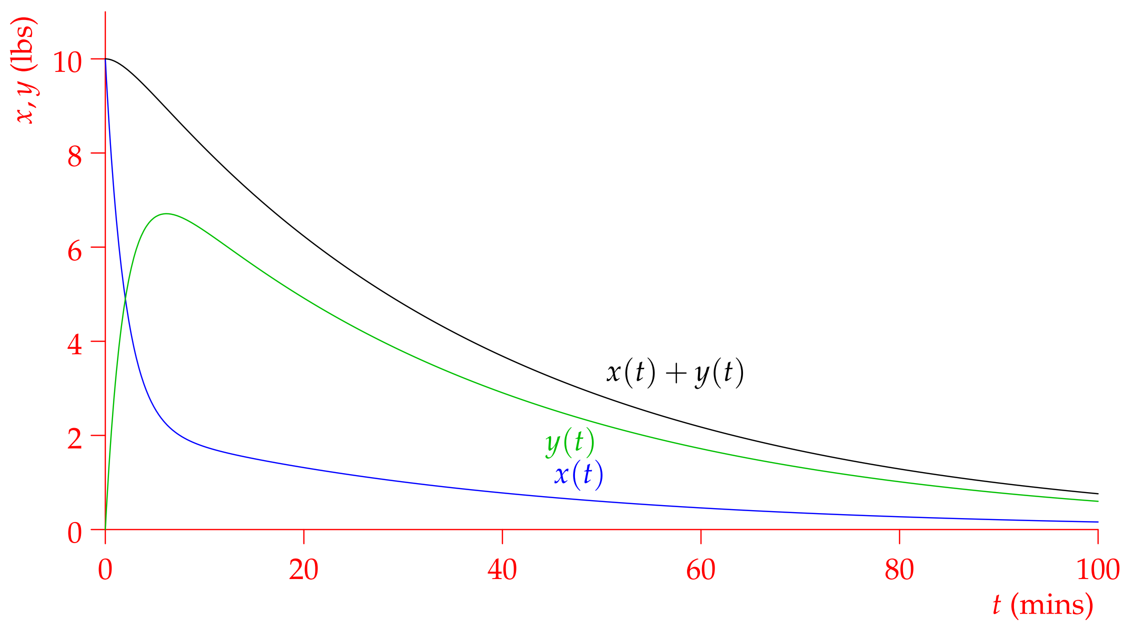
which yield

$$a = \frac{5}{\sqrt{13}}(\sqrt{13}-2), \qquad b = \frac{5}{\sqrt{13}}(\sqrt{13}+2),$$

for solutions

$$\begin{cases} x(t) = \frac{5}{\sqrt{13}} e^{-\frac{4}{15}t} \left((\sqrt{13}-2)e^{\frac{\sqrt{13}}{15}t} + (\sqrt{13}+2)e^{-\frac{\sqrt{13}}{15}t} \right) \\ y(t) = \frac{30}{\sqrt{13}} e^{-\frac{4}{15}t} \left(e^{\frac{\sqrt{13}}{15}t} - e^{-\frac{\sqrt{13}}{15}t} \right) \end{cases}$$

The plots are graphed below, along with the total mass of salt in the entire system:



The salt is maximal in tank y at the only critical point of $y(t)$:

$$\begin{aligned} y'(t) &= \frac{30}{\sqrt{13}} \left(\frac{\sqrt{13}-4}{15} e^{\frac{\sqrt{13}-4}{15}t} + \frac{\sqrt{13}+4}{15} e^{-\frac{\sqrt{13}+4}{15}t} \right) \\ &= \frac{30}{15\sqrt{13}} e^{-\frac{\sqrt{13}+4}{15}t} \left((\sqrt{13}-4) e^{\frac{2\sqrt{13}}{15}t} + \sqrt{13}+4 \right) \\ &= 0 \iff t = \frac{15}{2\sqrt{13}} \ln \frac{4+\sqrt{13}}{4-\sqrt{13}} \approx 6.155 \text{ mins} \end{aligned}$$

with $y_{\max} \approx 6.71$ lb.