Chaos in Dynamical Systems — non-examinable

Differential equations are *deterministic*[1]

A basic (yet false!) assumption: trajectories starting close together should stay close for all time

Non-linear dynamical systems often exhibit chaotic behavior: some (or all!) nearby trajectories may eventually end up behaving wildly different

Similar to the famous 'butterfly effect'

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[1] Given perfect information about initial conditions and infinite computing power, should be able to determine with any accuracy necessary the evolution of the solution of a system of ODEs

Population growth and period doubling

Consider logistic equation with stable population $M = 1$:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1-P) \qquad k > 0$$

Solution $P(t) = \frac{P_0}{P_0+(k-P_0)e^{-kt}}$ approaches $P = 1$ for any $P_0 > 0$

Use Euler's method with step-size h:

$$t_{n+1} = t_n + h \qquad P_{n+1} = P_n + khP_n(1-P_n)$$

$$\therefore P_{n+1} = (1+kh)P_n\left(1 - \frac{kh}{1+kh}P_n\right)$$

Let $P_n = \frac{1+kh}{kh}x_n$ to obtain the difference equation

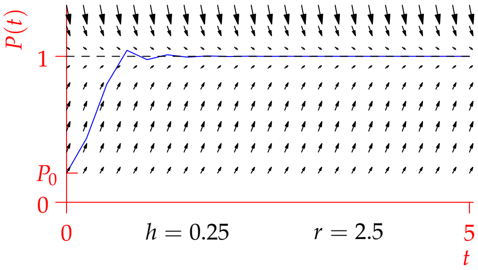
$$x_{n+1} = rx_n(1-x_n) \qquad r = 1 + kh > 1$$

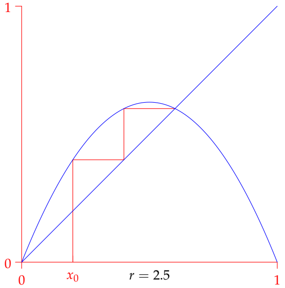
Iterating as $n \to \infty$ should yield $\frac{kh}{1+kh} = \frac{r-1}{r} = 1 - r^{-1}$

However, this only occurs for relatively small values of $r\ldots$

- $x_n \to 1 - r^{-1}$ for $r \leq 3$
- $x_n \to$ period 2 cycle for $3 < r \lessapprox 3.45$
- $x_n \to$ period 4 cycle for $3.45 \lessapprox r \lessapprox 3.54$
- Chaos: $r \approx 3.57$

Animations[2] show first 1000 iterations for various values of r
$y = x$ and $y = rx(1 - x)$ in blue: intersection $x = y = 1 - r^{-1}$





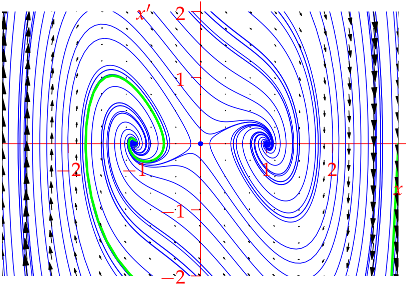
[2] For more see the `Logistic Map`

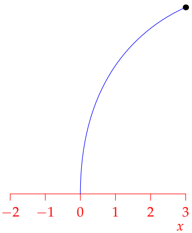
The Duffing Equation

Imagine a mass balanced on a flexible filament: modeled by

$$x'' + x' - x + x^3 = 0$$

where $x(t)$ is deviation to the right at time t
With $y = x'$ have critical points $(x, y) = (0,0), (1,0), (-1,0)$
Linearization $\Longrightarrow$ $(0,0)$ saddle, $(\pm 1, 0)$ stable spirals
For almost all initial conditions mass moves towards $x = \pm 1$



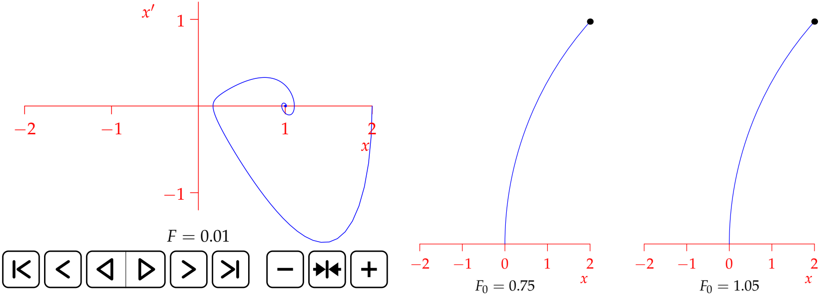


The Forced Duffing Equation

Apply periodic force to filament:

$$x'' + x' - x + x^3 = F_0 \cos t$$

As F_0 increases, critical points become foci of *attractors*: trajectories approach particular paths as $t \to \infty$



$0 < F_0 \lessapprox 0.67$: attractors simple closed loops
$0.67 \lessapprox F_0$: period-doubling $\rightsquigarrow$ chaos
$1 \lessapprox F_0 \lessapprox 1.05$: periodic attractor before more chaos

The Lorenz Attractor

The Lorenz system of differential equations is

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma x + \sigma y \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \rho x - y - xz \\ \frac{\mathrm{d}z}{\mathrm{d}t} = -\beta z + xy \end{cases}$$

where σ, β, ρ are constants: highly simplified model for atmospheric conditions in weather forecasting.

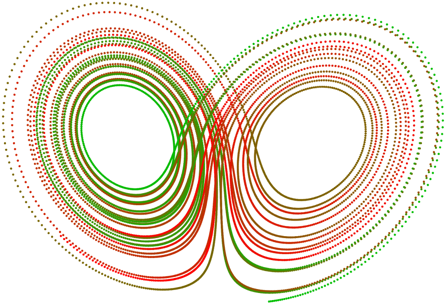
Solution curve (in 3D):
`Lorenz strange attractor`

Plot has $\beta = 8/3$, $\sigma = 10$, $\rho = 28$,

$\mathbf{x}_0 = (4, 5, 6)$, $0 \leq t \leq 30$ seconds,

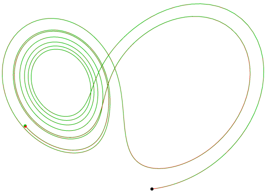
10,000 Runge–Kutta iterations,

Green $\rightarrow$ red as t increases

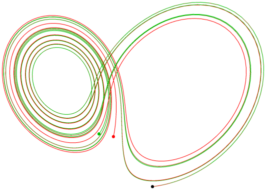


Seemingly random jumps from one set of loops to the other. System widely studied, yet much unexplained behavior remains:

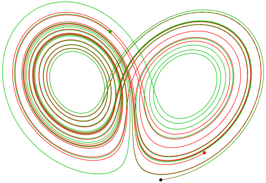
1. Even if number of time a trajectory loops one side of the attractor is known, number of further loops before trajectory jumps to the other side is unknown.
2. Very fine dependence on initial conditions: e.g. for initial conditions $\mathbf{x}_0 = (4, 5, 6)$ and $(4.001, 5, 6)$, trajectories almost identical for ≈ 13 seconds, before suddenly separating.



6.5 seconds



13 seconds



19.5 seconds