6.3 Convolutions

Example Consider the initial value problem

$$\begin{cases} x' + 3x = f(t) \\ x(0) = a \end{cases}$$

where $f(t)$ is an undecided function, and a is constant.

(a) Solve the problem using the intgerating factor method.

(b) Solve the problem using Laplace transforms.

Solution

1. The integrating factor is $I(t) = \exp\left(\int 3\,\mathrm{d}t\right) = e^{3t}$, whence the solution is the integral

   $$x(t) = \frac{1}{e^{3t}}\left[\int_0^t e^{3\tau} f(\tau)\,\mathrm{d}\tau + a\right]$$

   where we use the definite integral to incorporate the initial condition. Note that we cannot evaluate the integral without knowing $f(t)$ explicitly.

2. Taking Laplace transforms, and writing $F(s) = \mathcal{L}\{f(t)\}$, we have

   $$sX(s) - x'(0) + 3X(s) = F(s) \Longrightarrow X(s) = \frac{F(s)+a}{s+3} = \mathcal{L}\left\{e^{-3t}\right\} \cdot \mathcal{L}\{f(t)\} + \mathcal{L}\left\{ae^{-3t}\right\}$$

   Using the convolution formula we can invert the transform:

   $$X(s) = \mathcal{L}\left\{e^{-3t} * f(t) + ae^{-3t}\right\}$$
   $$\Longrightarrow x(t) = \int_0^t e^{-3(t-\tau)} f(\tau)\,\mathrm{d}\tau + ae^{-3t} = \frac{1}{e^{3t}}\left[\int_0^t e^{3\tau} f(\tau)\,\mathrm{d}\tau + a\right]$$

   exactly as in part (a).

Note: don't use the Laplace transform method unless you're forced to. Method 1 is much easier!