1.2 Existence and Uniqueness (Picard's Theorem)

Three examples where the theorem *does not apply*.

1. The initial value problem

$$\frac{dy}{dx} = \frac{1}{1-x} \quad y(1) = 1$$

fails to satisfy Picard's Theorem since $f(x,y) = \frac{1}{1-x}$ is not defined (let alone continuous) at $(x,y) = (1,1)$. In fact this IVP has no solutions.[1]

2. The initial value problem

$$\frac{dy}{dx} = \frac{1}{2y} \quad y(0) = 0$$

fails Picard's Theorem since $f(x,y) = \frac{1}{2y}$ is not defined at the initial condition $(x,y) = (0,0)$ (so is not continuous there). This IVP has *two solutions* $y(x) = \pm\sqrt{x}$. This may feel deceptive, since we cannot differentiate $\sqrt{x}$ at $x = 0$!

3. For the initial value problem

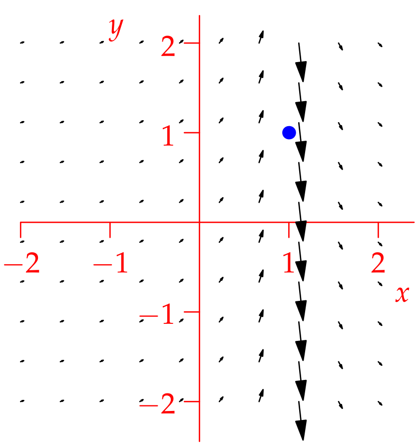
$$\frac{dy}{dx} = \frac{3}{2}y^{1/3} \quad y(0) = 0$$

the right side $f(x,y) = \frac{3}{2}y^{1/3}$ is continuous at $(x,y) = (0,0)$, but its y-derivative is not:

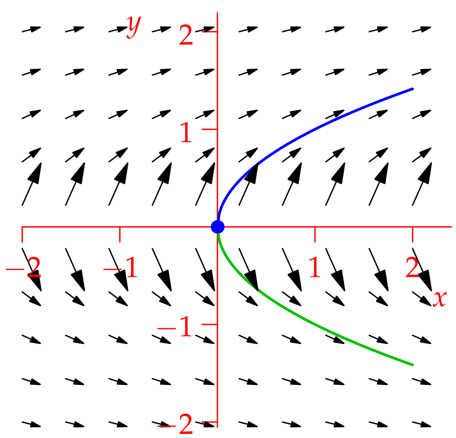
$$\frac{\partial f}{\partial y} = \frac{1}{2}y^{-2/3}$$

The IVP therefore fails Picard's Theorem. In fact this problem has *infinitely many solutions*: $y(x) \equiv 0$ is a solution, as is any function of the form

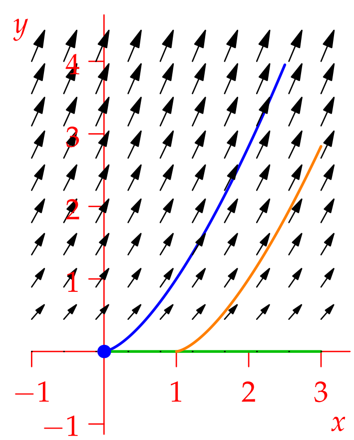
$$y(x) = \begin{cases} 0 & x < a \\ (x-a)^{3/2} & x \geq a \end{cases} \quad \text{for any constant } a \geq 0$$



Ex 1: no solutions



Ex 2: two solutions



Ex 3: infinitely many solutions

[1]This is a little tricky: if there were a solution on some interval $[0,t)$, say, then the solution would have the form $y = C - \ln(1-x)$ for some constant C. But this has a vertical asymptote at $x = 1$...