Math 3D: Differential Equations  
Final Exam (44520)  
Dec 9th, 2016  
13:30–15:30

Name:  
Student Id#:  

Total marks = 100 (per question in brackets)  
No calculators or other electronic devices  
Unless otherwise stated, include all your working for full credit  
Try all parts of every question, even if you can’t do the first part!

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1. (a) Find the solutions to the following initial value problems:
   
   i. \[
   \frac{dy}{dx} + 2y = 1, \quad y(0) = 2
   \]

   ii. \[
   \frac{dy}{dx} = y(y + 1), \quad y(0) = 3
   \]
(b) Explain how you know that \( y(x) \equiv -1 \) is the only solution to the initial value problem

\[
\frac{dy}{dx} = y(y + 1), \quad y(0) = -1
\]

2. A linear constant coefficient differential equation has the following general solution:

\[ y(x) = c_1 e^x + c_2 xe^x + c_3 \cos 2x + c_4 \sin 2x \]

What is the differential equation?
3. Find the general solution to the linear ODE

\[ x''' - 3x' + 2x = e^{-t} \]
4. Consider the spring system \( x'' + 4x' + 3x = 0 \)

(a) Is the system over-, under- or critically damped? (3)

(b) Find the impulse response: otherwise said, find the solution to

\[
\begin{align*}
  x'' + 4x' + 3x &= \delta(t) \\
  x(0) &= 0 = x'(0)
\end{align*}
\]

(c) Use your answer to part (b) to write down the solution to

\[
\begin{align*}
  x'' + 4x' + 3x &= \sin(t^2) \\
  x(0) &= 0 = x'(0)
\end{align*}
\]

as an integral. Do not attempt to evaluate the integral. (4)
5. Bessel’s Equation of order $\frac{1}{3}$ is

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

(a) Show that $x = 0$ is a regular singular point, and that the roots of the indicial equation are $r = \pm \frac{1}{3}$.

(b) Apply the method of Frobenius to show that a series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n-1/3}$$

solves Bessel’s equation if and only if the coefficients satisfy the relations

$$\begin{cases} 
  a_1 = 0 \\
  a_n = \frac{-3}{n(3n-2)} a_{n-2} \quad \text{for all } n \geq 2
\end{cases}$$
(c) Find the first three non-zero terms of the solution corresponding to \( r = -\frac{1}{3} \)  \( (5) \)
6. The populations of two co-operating species are modeled by the nonlinear system

\[
\frac{dx}{dt} = f(x, y) = x(1 - x + ky) \quad \frac{dy}{dt} = g(x, y) = y(1 + kx - y)
\]

where \(k \geq 0\) is constant

(a) Suppose that there is no co-operation \((k = 0)\) so that both populations are independent and obey a logistic equation. If the initial conditions \(x(0), y(0)\) of both populations are positive, what happens to the populations in the long term? Why? (5)

(b) Suppose that \(k = \frac{1}{2}\) so that the co-operation rate is low. Compute the Jacobian \(\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\) and use it to identify the types of the four critical points \((0, 0), (1, 0), (0, 1), (2, 2)\). (10)
(c) Continuing to assume that $k = \frac{1}{2}$, roughly sketch the direction field. What do you expect to happen to the populations $x(t), y(t)$ as $t \to +\infty$. 

(d) Suppose now that the co-operation rate is $k = 1$. What do you expect to happen to the populations as $t \to +\infty$? Explain. (No calculation is required)