Math 3D: Differential Equations

Final Exam (v1) (44660)

March 16th 2015
10.30am–12.30pm

Name:

Student Id#:

Total marks = 100 (per question in brackets)
No calculators or other electronic devices
Unless otherwise stated, include all your working for full credit
Try all parts of every question, even if you can’t do the first part!

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1. Find the solutions to the following differential equations using whatever method you like.

(a) \( \frac{dy}{dx} = -\frac{1}{(x - 4)^2} \)  

(b) \( \frac{dy}{dx} = -2x^2y^3, \quad y(1) = \frac{1}{2} \)
2. State Picard’s Theorem and use it to explain why the initial value problem
\[
\begin{cases}
\frac{dy}{dx} = \frac{x}{y-2} + e^x \\
y(3) = 1
\end{cases}
\]
has a unique solution.

(c) \( 2x'' + 5x' - 3x = 11 - 3t \)
3. You jump out of a plane and immediately open your parachute: your downward velocity \( v(t) \geq 0 \) (feet per second) satisfies the initial value problem

\[
\begin{cases}
\frac{dv}{dt} = \frac{8}{5}(20 - v) \\
v(0) = 0
\end{cases}
\]

(a) Sketch the slope field for this differential equation and sketch the solution. (5)

(b) Solve for \( v(t) \). (5)

(c) How long does it take to reach 95% of your limiting velocity? Given that \( e^3 \approx 20 \), round your answer to the nearest second. (5)
4. A spring satisfies the initial value problem

\[
\begin{align*}
    x'' + 6x' + 9x &= 0 \\
    x(0) &= 1 \\
    x'(0) &= a
\end{align*}
\]

where \(a\) is a constant.

(a) Is the spring under-, over-, or critically-damped? \(\text{(2)}\)

(b) Suppose that the mass passes through the equilibrium point \(x = 0\) meters at \(t = 2\) seconds. Find the initial speed of the mass \(a\). \(\text{(8)}\)
5. (a) Use the eigenvalue/eigenvector method to find the general solution to the matrix differential equation

\[ x' = \begin{pmatrix} 3 & -3 \\ 0 & -2 \end{pmatrix} x \]

(b) Consider the non-linear system

\[ \begin{align*}
\frac{dx}{dt} &= x^2 - xy - 3x \\
\frac{dy}{dt} &= xy - 5y
\end{align*} \]

Find the three critical points and identify their types.

(part(c) next page)
(c) Use your answer to part (b) to sketch the slope field for the system.
6. A 1kg-mass on a spring satisfies the initial value problem

\[
\begin{cases}
x'' + 4x = 8\delta(t - 2\pi) \\
x(0) = 3, \quad x'(0) = 0
\end{cases}
\]

(a) Suppose that time is measured in seconds and distance in meters. What happens to the velocity of the mass at \( t = 2\pi \)?

(b) Show that the Laplace transform of the solution \( x(t) \) is given by

\[
X(s) = \frac{3s + 8e^{-2\pi s}}{s^2 + 4}
\]
(c) Find the solution to the problem and sketch it. In particular, what is the amplitude of the periodic motion after \( t = 2\pi \) seconds?
7. Consider the differential equation

\[ 2x^2y'' + 3xy' - (x^2 + 1)y = 0 \]

(a) Show that \( x = 0 \) is a regular singular point of this ODE and that the roots of the indicial equation are \( r_1 = \frac{1}{2} \) and \( r_2 = -1 \)

(b) By part (a), there are two independent solutions

\[ y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}, \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1} \]

Obtain a recurrence relation for the coefficients \( a_n \) of the first solution and, supposing that \( a_0 = 1 \), find the first three non-zero terms of the solution \( y_1(x) \).