## General/Singular Solution Example (1.3 Separable Equations)

**Example** Consider the 1[st] order ODE $\frac{dy}{dx} = x^2 y(y-1)$

1. Find two constant solutions to the ODE.
2. Find a general solution using separation of variables.
   (*One of the two constant solutions will be singular*).
3. Find an alternative general solution so that the *other* constant solution is singular.

**Solution**

1. $y(x) \equiv 0$ and $y(x) \equiv 1$ both solve the ODE.
2. Separate variables to obtain (note the partial fraction step...)

$$\begin{aligned} \int \frac{1}{y(y-1)}\,dy = \int x\,dx \implies & \int \frac{1}{y-1} - \frac{1}{y}\,dy = \int x^2\,dx && \text{(partial fractions!)} \\ \implies & \ln\left|\frac{y-1}{y}\right| = \frac{1}{3}x^3 + c \\ \implies & \frac{y-1}{y} = Ae^{\frac{x^3}{3}} && (*) \\ \implies & y(x) = \frac{1}{1 - Ae^{\frac{x^3}{3}}} \end{aligned}$$

   Choosing $A = 0$ results in the constant solution $y(x) \equiv 1$. There is no way to choose the constant $A$ in such a manner that $y(x) \equiv 0$ is encompassed by this general solution: $y(x) \equiv 0$ is singular.

3. We could instead have written $\alpha = \frac{1}{A}$ in step $(*)$, giving a new general solution:

$$y(x) = \frac{\alpha}{\alpha - e^{\frac{x^3}{3}}} = \frac{\alpha e^{-\frac{x^3}{3}}}{\alpha e^{-\frac{x^3}{3}} - 1}$$

   This time, choosing $\alpha = 0$ yields the other constant solution $y(x) \equiv 0$. We cannot choose $\alpha$ in such a way as to obtain the constant solution $y(x) \equiv 1$, which is therefore singular *for this general solution*.

The Moral of the Story is that singular solutions are singular/special *relative to some general solution*.