1.3 Separable Equations

Example. Consider the equation $\frac{dy}{dx} = x^2 y(y-1)$

(a) Find two constant solutions to the ODE.

(b) Find a general solution using separation of varaiables: note that *one* of the constant solutions is singular.

(c) Write the solution in another way so that the *other* constant solution is singular.

Solution. (a) $y(x) \equiv 0$ and $y(x) \equiv 1$ both solve the ODE.

(b) Separating variables and solving, we obtain (note the partial fraction step...)

$$\int \frac{1}{y(y-1)}\, dy = \int x\, dx \Longrightarrow \int \frac{1}{y-1} - \frac{1}{y}\, dy = \int x^2\, dx$$
$$\Longrightarrow \ln\left|\frac{y-1}{y}\right| = \frac{1}{3}x^3 + c$$
$$\Longrightarrow \frac{y-1}{y} = Ae^{\frac{x^3}{3}}$$
$$\Longrightarrow y(x) = \frac{1}{1 - Ae^{\frac{x^3}{3}}}$$

Choosing $A = 0$ yields the constant solution $y(x) \equiv 1$. There is no way to choose A such that $y(x) \equiv 0$ is encompassed by this general solution; thus $y(x) \equiv 0$ is singular.

(c) Could instead let $\alpha = \frac{1}{A}$, giving a new general solution:

$$y(x) = \frac{\alpha}{\alpha - e^{\frac{x^3}{3}}} = \frac{\alpha e^{-\frac{x^3}{3}}}{\alpha e^{-\frac{x^3}{3}} - 1}$$

Now choosing $\alpha = 0$ yields the constant solution $y(x) \equiv 0$. There is now no way to choose α in order to obtain the constant solution $y(x) \equiv 1$, which is thus singular *for this general solution*.

Moral of the story: singular solutions are singular/special *relative to some general solution*.