Math 3D Differential Equations Homework Questions 1

1. Write a differential equation that is a mathematical model of the following situation:

   The acceleration $\frac{\mathrm{d}v}{\mathrm{d}t}$ of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

2. (a) If k is a constant, show that a general (one-parameter) solution of the differential equation

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$$

   is given by $x(t) = \frac{1}{c-kt}$, where c is an arbitrary constant.

   (b) Determine by inspection a solution of the initial value problem

   $$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = kx^2 \\ x(0) = 0 \end{cases}$$

3. Suppose that the velocity of a motorboat coasting in water satisfies the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = kv^2$. The initial speed of the motorboat is $v(0) = 10\ \mathrm{ms}^{-1}$, and v is decreasing at a rate of $1\ \mathrm{ms}^{-2}$ when $v = 5\ \mathrm{ms}^{-1}$.

   (a) How long does it take for the velocity of the boat to decrease to $1\ \mathrm{ms}^{-1}$?

   (b) To $\frac{1}{10}\ \mathrm{ms}^{-1}$?

   (c) When does the boat come to a stop, or does it?

4. Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + x$ with $y(1) = 3$

5. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2-1}$ for $y(0) = 0$

6. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y+1}$ for $y(0) = 0$

7. Find a function $y = f(x)$ satisfying the initial value problem

   $$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x+2}} \\ y(2) = -1 \end{cases}$$

8. A car traveling at 60 miles/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

9. Arthur C. Clarke's *The Wind from the Sun* describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.001g = 0.0098\ \mathrm{ms}^{-2}$. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth the speed of light $c = 3 \times 10^8\ \mathrm{ms}^{-1}$. How long will it take the spacecraft to catch up with the projectile?

10. For each initial value problem, determine whether Picard's Theorem guarantees the existence and uniqueness of a solution.

   (a) $\frac{dy}{dx} = 2x^2y^2$, $y(1) = -1$
   (b) $\frac{dy}{dx} = \sqrt{x-y}$, $y(2) = 2$
   (c) $y\frac{dy}{dx} = x - 1$, $y(0) = 1$

11. (a) Verify that if $c \geq 0$ is constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leq c \\ (x-c)^2 & \text{for } x > c \end{cases}$$

   satisfies the differential equation $y' = 2\sqrt{y}$ for all x (including the point $x = c$). Construct a figure illustrating the fact that the initial value problem

$$\begin{cases} y' = 2\sqrt{y} \\ y(0) = 0 \end{cases}$$

   has infinitely many different solutions.

   (b) For what values of b does the initial value problem

$$\begin{cases} y' = 2\sqrt{y} \\ y(0) = b \end{cases}$$

   have (i) no solution, (ii) a unique solution that is defined for all x?

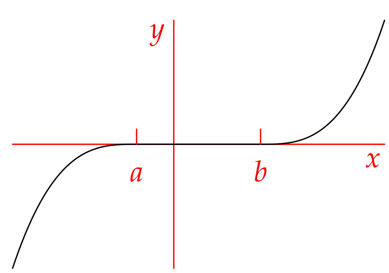
12. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem

$$\begin{cases} xy' = y \\ y(a) = b \end{cases}$$

   has—one, none, or infinitely many.

13. (a) Let $a \leq b$ be constants. Show that the following function satisfies the differential equation $\frac{dy}{dx} = 5y^{4/5}$.

$$y(x) = \begin{cases} (x-a)^5 & \text{if } x < a, \\ 0 & \text{if } a \leq x \leq b, \\ (x-b)^5 & \text{if } x > b, \end{cases}$$



   (b) Find two distinct solutions to the ODE which satisfy the initial condition $y(1) = 1$.

   (c) $5y^{4/5}$ is continuous at $(1, 1)$, as is its y-derivative $4y^{-1/5}$. How can you reconcile your observation in part (b) with Picard's Existence/Uniqueness Theorem?

14. For each of the following equations, find general solutions. These may be implicit if necessary.

   (a) $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^2 = 0$

   (b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{xy}$

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-1)y^5}{x^2(2y^3-y)}$

15. Solve $\frac{\mathrm{d}x}{\mathrm{d}t} = (x^2-1)t$ for $x(0) = 0$.

16. (a) Find a general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$.

   (b) Find a singular solution that is not included in the general solution.

   (c) Suppose values a, b are given. Show that the initial value problem

$$\begin{cases} y' = y^2 \\ y(a) = b \end{cases}$$

   has a unique solution and state what it is (in terms of a, b).

17. A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

   (*Hint: Use Newton's Law of cooling* $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_s)$, *where* $T(t)$ *is the temperature at time* t, T_s *is the temperature of the surroundings, and* k *is a constant*...)