Math 3D Differential Equations Homework Questions 2

1. Find the general solution of the differential equation. If an initial condition is given, find the corresponding particular solution.

   (a) $y' + 3y = 2xe^{-3x}$
   (b) $\frac{dy}{dx} + 6y = e^x$
   (c) $\frac{dy}{dx} + xy = x$
   (d) $y' + 5y = \sin x$, $y(0) = 0$
   (e) $xy' + 5y = 7x^2$, $y(2) = 5$
   (f) $(x^2 + 4)y' + 3xy = x$, $y(0) = 1$

2. Express the solution of the initial value problem

$$\begin{cases} 2x\frac{dy}{dx} = y + 2x\cos x \\ y(1) = 0 \end{cases}$$

   as an integral

3. Consider a cascade of two water tanks. Fresh water flows into tank 1. Well-mixed salty water flows out of tank 1 into tank 2, and out of tank 2. At time t, each tank has

Tank	Volume	Inflow rate	Outflow rate	Quantity of Salt
1	$V_1 = 100$ gal	5 gal/min	5 gal/min	$x(t)$ lb
2	$V_2 = 200$ gal	5 gal/min	5 gal/min	$y(t)$ lb

   Each tank initially contains 50 lb of salt.

   (a) Find the quantity $x(t)$ of salt in tank 1 at time t.
   (b) Show that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$$

   and then solve for $y(t)$, using the function $x(t)$ found in part (a).
   (c) Finally, find the maximum quantity of salt in tank 2.

4. Solve $f(x) = 0$ to find the critical points of the given autonomous differential equation $\frac{dx}{dt} = f(x)$. Analyze the sign of $f(x)$ to determine whether each critical point is stable or unstable, and construct the phase diagram. Solve the differential equation explicitly for $x(t)$ in terms of t. Finally sketch the slope field and several typical solution curves.

   (a) $\frac{dx}{dt} = 3x - x^2$
   (b) $\frac{dx}{dt} = -(3 - x)^2$

5. Find the critical points for the differential equation $\frac{dx}{dt} = x^3(x^2 - 4)$ and determine whether they are stable or unstable. Sketch several solution curves. *You do not have to solve the equation!*

The remaining questions are on population models. Solutions to the various population models (such as the logistic equation) should be treated as open-book.

6. A prolific breed of rabbits have constant birth and death rates, where $\beta - \delta = kP > 0$ is proportional to the rabbit population $P = P(t)$.

   Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. How long does it take for the population to "explode"?

7. Suppose that the population of a country $P(t)$ (million) satisfies the logistic differential equation

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(200 - P)$$

   where k is constant. Its population in 1940 was 100 million, at which time it was growing at 1 million per year. Predict the country's population in the year 2000.

8. (An example from Chemistry) Potassium nitrate KNO_3 dissolves in methanol: this is modelled such that the mass (x grams) dissolved at time t seconds satisfies the differential equation

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.8x - 0.004x^2$$

   (a) What is the maximum quantity of potassium nitrate that will stay in solution?
   In accordance with the logistic model, if we somehow managed to dissolve more than the maximum, the extra would immediately start to precipitate out of solution.

   (b) If $x(0) = 50\,\mathrm{g}$, how long will it take for an additional $50\,\mathrm{g}$ to dissolve?

9. Birth and death rates of animal populations are not typically constant, instead varying periodically with the passage of the seasons.

   (a) Find $P(t)$ if the population satisfies

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = (k + b\cos 2\pi t)P$$

   where t is measured in years, and k and b are positive constants.

   (b) The growth rate function $r(t) = k + b\cos 2\pi t$ varies periodically about its mean value k. Construct a graph that contrasts the growth of this population with one that has the same initial value P_0 but satisfies the natural growth equation $P' = kP$, for the same constant k.

10. (Hard) We generalize the *fish-harvesting* example. Suppose the stable population of fish is M and that h fish are harvested per unit time, resulting in the modified logistic equation

    $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(M - P) - h$$

    (a) Find the roots P_1, P_2 of the equation $kP(M - P) - h = 0$ using the quadratic formula.
    (*The answers depend on k, M and h*)

    (b) There are three generic possibilities for the roots in part (a). In each case, describe behavior of the fish population, and how it depends on the initial population P_0:

    i. $P_1 > P_2$ are both real numbers.
    ii. $P_2 = P_2$ is a real number.
    iii. P_1, P_2 are non-real complex numbers.