Math 3D Differential Equations Pre-Midterm 1 Exercises

1. A homogeneous second-order linear differential equation, two functions y_1, y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions.

   (a) $y'' - 9y = 0; \quad y_1 = e^{3x}, y_2 = e^{-3x}; \quad y(0) = -1, y'(0) = 15$

   (b) $y'' + 25y = 0; \quad y_1 = \cos(5x), y_2 = \sin(5x); \quad y(0) = 10, y'(0) = -10$

   (c) $y'' - 10y' + 25y = 0; \quad y_1 = e^{5x}, y_2 = xe^{5x}; \quad y(0) = 3, y'(0) = 13$

   (d) $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2, y_2 = x^{-3}; \quad y(2) = 10, y'(2) = 15$

2. Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution. *Thus the superposition principle does not generally hold for non-linear equations.*

3. A homogeneous second-order linear ODE with constant coefficients is given. Find the general solution.

   (a) $2y'' + 2y' - 4y = 0$

   (b) $y'' + 9y' - 10y = 0$

   (c) $y'' - 8y' + 16y = 0$

   (d) $y'' + 6y' + 13y = 0$

4. Show that the functions $f(x) = 17, \quad g(x) = 2\sin^2 x, \quad h(x) = 3\cos^2 x$ are linearly dependent on the real line. That is, find a non-trivial linear combination of f, g, h that vanishes identically.

5. Three independent solutions are given to the following differential equation. Find the solution satisfying the given initial conditions.

$$y''' - 3y'' + 4y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$$
$$y_1 = e^x, \quad y_2 = e^x \cos x, \quad y_3 = e^x \sin x$$

6. A body with mass 250 g is attached to the end of a spring that is stretched 20 cm by a force of 9 N. At time $t = 0$, the body is pulled 1 m to the right, stretching the spring, and set in motion with an initial velocity of 5 ms^{-1} to the left.

   (a) Find the spring constant k.

   (b) Find $x(t)$.

   (c) Find the amplitude and period of motion of the body

7. (Hard) Most grandfather clocks have pendulums with adjustable lengths. One such clock loses 10 min per day when the length of its pendulum is 30 in. With what length of pendulum will this clock keep perfect time?

   Assume that the differential equation for a pendulum of length L is $L\theta'' + g\theta = 0$*, with* $g = 32\ \text{ft}\,\text{s}^{-2}$

8. Suppose you have a damped spring system

$$2x'' + 16x' + 40x = 0, \qquad x(0) = 5, x'(0) = 4$$

(a) Determine whether the system is over-, under-, or critically damped

(b) Find the motion $x(t)$ when the damper is disconnected ($2x'' + 40x = 0$)

(c) Solve for the motion $x(t)$ when the damper is connected and sketch the solution on the same graph as the solution to (b), indicating how the damping changes the solution.

9. A non-homogeneous differential equation, a complementary solution y_C, and a particular solution y_P are given. Find a solution satisfying the given initial conditions.

$$y'' - 2y' + 2y = 2x; \quad y(0) = 4, \quad y'(0) = 8$$
$$y_C = c_1 e^x \cos x + c_2 e^x \sin x; \quad y_P = x + 1$$

10. (a) Find by inspection particular solutions of the two non-homogeneous equations

$$y'' + 2y = 4 \quad \text{and} \quad y'' + 2y = 6x$$

(b) Use your answer to part (a) to find a solution to the non-homogeneous equation

$$y'' + 2y = 4 + 6x$$

11. Find a particular solution y_P. Primes denote derivatives with respect to x.

(a) $2y'' + 4y' + 7y = x^2$

(b) $y'' + 9y = 2x^2 e^{3x} + 5$

12. Find the general solution to the following ODEs

(a) $y'' + 5y' + 6y = 2x + 1$

(b) $y'' - y' + y = 2\sin(3x)$