

Math 3D Differential Equations Homework Questions 3

1. Solve the initial value problem.

(a) $y'' + y = \cos x, \quad y(0) = 1, y'(0) = -1$

(b) $y'' + 2y' + y = \sin 3x, \quad y(0) = 2, y'(0) = 0$

2. (a) Euler's formula tells us that

$$\cos 3x + i \sin 3x = e^{3ix} = (\cos x + i \sin x)^3$$

Expand and equate real and imaginary parts to derive the trigonometric identities

$$\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x, \quad \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

(b) Use the result of part (b) to find a general solution of

$$y'' + 4y = \cos^3 x$$

(c) A mass on a spring without damping is acted on by the external force $F(t) = F_0 \cos^3 \omega t$. Show that there are *two* values of ω for which resonance occurs, and find both.

3. Find the steady periodic solution $x_{\text{sp}}(t) = C \cos(\omega t - \alpha)$ of the equation

$$x'' + 3x' + 5x = -4 \cos 5t$$

Graph $x_{\text{sp}}(t)$ together with the adjusted forcing function $F_1(t) = \frac{F(t)}{m\omega^2} = -\frac{4}{5} \cos 5t$.

4. Find and plot both the steady periodic function and the transient solution $x_{\text{tr}}(t)$ that satisfies the given initial value problem.

(a) $x'' + 6x' + 13x = 10 \sin 5t; \quad x(0) = x'(0) = 0$

(b) $x'' + 8x' + 25x = 200 \cos t + 520 \sin t; \quad x(0) = -30, x'(0) = -10$

5. Transform the following system of equations into an equivalent system of first-order differential equations

$$\begin{cases} x'' - 5x + 4y = 0 \\ y'' + 4x - 5y = 0 \end{cases}$$

6. Use the method of substitution/elimination to solve the given initial value problem

(a) $\begin{cases} x' = 10y \\ y' = -10x \end{cases} \quad x(0) = 3, \quad y(0) = 4$

(b) $\begin{cases} x' = -y \\ y' = 10x - 7y \end{cases} \quad x(0) = 2, \quad y(0) = -7$

(c) $\begin{cases} x' = x + 9y \\ y' = -2x - 5y \end{cases} \quad x(0) = 3, \quad y(0) = 2$

7. (a) Take the solution to question 6a and calculate $x^2 + y^2$ to show that the trajectory is a circle.
 (b) Show similarly that the trajectories of the system $x' = \frac{1}{2}y$, $y' = -8x$ are ellipses with equations of the form $16x^2 + y^2 = C^2$.
8. (Hard) Three 100 gallon brine tanks are connected, so that the quantities of brine in each $(x_1(t), x_2(t), x_3(t))$ lb) satisfy

$$\begin{cases} 10x'_1 = x_3 - x_1 \\ 10x'_2 = x_1 - x_2 \\ 10x'_3 = x_2 - x_3 \end{cases}$$

Assume that the first tank contains 100 lb of salt, whereas the other two are filled with fresh water. Find the amount of salt in each of the three tanks at time t .

Hint: Why must $x_1 + x_2 + x_3$ be constant?