Math 3D Differential Equations Pre-Midterm 2 Exercises

1. Solve the linear system and decide whether the critical point $(0,0)$ is stable or unstable. Sketch the direction field and use it to decide whether $(0,0)$ is a node, a center, a saddle or a spiral point.

   (a) $\begin{cases} x' = -y \\ y' = 4x \end{cases}$ (b) $\begin{cases} x' = y \\ y' = -5x - 4y \end{cases}$

2. Identify the type of the critical point $(0,0)$ of the given system.

   (a) $\frac{dx}{dt} = 3x + y \qquad \frac{dy}{dt} = 5x - y$

   (b) $\frac{dx}{dt} = x - 2y \qquad \frac{dy}{dt} = 5x - y$

3. Each system has a unique critical point $(x_0, y_0) \neq (0,0)$. Find it and identify its type.

   (a) $\frac{dx}{dt} = x + y - 7 \qquad \frac{dy}{dt} = 3x - y - 5$

   (b) $\frac{dx}{dt} = x - 2y + 1 \qquad \frac{dy}{dt} = x + 3y - 9$

4. This question shows how small changes in a system can change the type of a critical point and/or its stability. Throughout, let ϵ be a given constant.

   (a) Consider the linear system

   $$\frac{dx}{dt} = \epsilon x - y \qquad \frac{dy}{dt} = x + \epsilon y$$

   Show that the critical point $(0,0)$ is:

   i. A stable spiral point if $\epsilon < 0$

   ii. A center if $\epsilon = 0$

   iii. An unstable spiral point if $\epsilon > 0$

   (b) Consider the linear system

   $$\frac{dx}{dt} = -x + \epsilon y \qquad \frac{dy}{dt} = x - y$$

   Show that the critical point $(0,0)$ is:

   i. A stable spiral point if $\epsilon < 0$

   ii. A stable node if $0 \leq \epsilon < 1$

   What happens if $\epsilon = 1$? If $\epsilon > 1$?

5. Solve the equation $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ to find the trajectories of the given system. Sketch the direction field and phase portrait, and decide whether $(0,0)$ is a node, a center, a saddle or a spiral point.

   (a) $\frac{dx}{dt} = y(1 + x^2 + y^2) \qquad \frac{dy}{dt} = x(1 + x^2 + y^2)$

   (b) $\frac{dx}{dt} = y^3 e^{x+y} \qquad \frac{dy}{dt} = -x^3 e^{x+y}$

6. Identify the type of the critical point $(0,0)$ of the given almost linear system. Are there any other critical points? Find them, but do not identify their type.

   (a) $\frac{dx}{dt} = 6x - 5y + x^2 \qquad \frac{dy}{dt} = 2x - y + y^2$

   (b) $\frac{dx}{dt} = 5x - 3y + y(x^2 + y^2) \qquad \frac{dy}{dt} = 5x + y(x^2 + y^2)$

7. Find all the critical points and identify the type and stability of each. Sketch a vector field of the system.

   (a) $\frac{dx}{dt} = y - 1 \qquad \frac{dy}{dt} = x^2 - y$

   (b) $\frac{dx}{dt} = xy - 2 \qquad \frac{dy}{dt} = x - 2y$

8. Separate the variables in the quotient

$$\frac{dy}{dx} = \frac{-150y + 2xy}{200x - 4xy}$$

   and thereby derive the exact implicit solution

$$200 \ln y + 150 \ln x - 2x - 4y = C$$

   of the system $x' = 200x - 4xy$, $y' = -150y + 2xy$.

9. Let $x(t)$ be a population of aphids that under natural conditions is held somewhat in check by a predator population $y(t)$ of ladybugs. Assume that $x(t)$ and $y(t)$ satisfy the standard predator-prey equations so that the equilibrium populations are

$$x_E = \frac{b}{q} \qquad y_E = \frac{a}{p}$$

   Now suppose that an insecticide is employed that kills (per unit time) the same fraction $f < a$ of each species of insect: what are the new equilibrium populations and how do they relate to x_E and y_E? Was the use of the insecticide a good idea?

10. The non-linear system

$$\frac{dx}{dt} = 60x - 4x^2 - 3xy \qquad \frac{dy}{dt} = 42y - 2y^2 - 3xy$$

   models two species in competition with each another. Show that the linearization at the critical point $(6,12)$ has eigenvalues $\lambda = -24 \pm 18\sqrt{2}$. What sort of critical point does the original system have? What does this mean about the potential of the species to coexist?

11. Consider the predator-prey system

$$\frac{dx}{dt} = x^2 - 2x - xy \qquad \frac{dy}{dt} = y^2 - 4y + xy$$

   (a) Show that the system has four critical points: $(0,0)$, $(0,4)$, $(2,0)$, and $(3,1)$

   (b) Linearize the system about each of the four critical points and identify each of their types. Use your answers to sketch a phase portrait for the system.