6.2 Transforms of Derivatives and ODEs

Theorem. *If $u(t-a) = u_a(t)$ is the step function centered at $t = a$, and $f(t)$ is any function whose Laplace transform is well-defined, then*

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

Proof. Just compute: note the change of limits on the integral because $u(t-a) = 0$ for $t < a$.

$$\begin{aligned}\mathcal{L}\{u(t-a)f(t-a)\} &= \int_0^\infty e^{-st}u(t-a)f(t-a)\,\mathrm{d}t = \int_a^\infty e^{-st}f(t-a)\,\mathrm{d}t \\ &= \int_0^\infty e^{-s(T+a)}f(T)\,\mathrm{d}T && \text{(substitute } T = t-a\text{)} \\ &= e^{-as}\int_0^\infty e^{-sT}f(T)\,\mathrm{d}T = e^{-as}\mathcal{L}\{f(t)\}\end{aligned}$$

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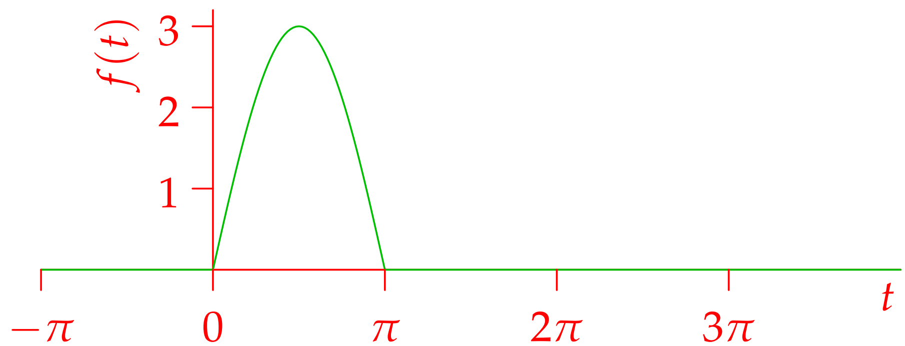
Example. Consider a spring driven by a function $f(t) = 3\sin t(1 - u(t-\pi))$: solve the initial value problem

$$x'' + 4x = f(t), \quad x(0) = x'(0) = 0$$

The force f is plotted below. We are assuming that $f(t) = 0$ for $t < 0$: this fits with the initial conditions of the motion, and the idea that Laplace transforms don't notice anything that happens before $t = 0$.



Solution. Writing $X(s) = \mathcal{L}\{x(t)\}$ for the Laplace transform of $x(t)$, we obtain

$$\begin{aligned}\mathcal{L}\{x''\} + 4\mathcal{L}\{x\} &= \mathcal{L}\{f(t)\} \\ s^2X - sx(0) - x'(0) + 4X &= 3\mathcal{L}\{\sin t\} - 3\mathcal{L}\{u(t-\pi)\sin t\} \\ s^2X + 4X &= \frac{3}{s^2+1} + 3\mathcal{L}\{u(t-\pi)\sin(t-\pi)\} && (\sin(t-\pi) = -\sin t) \\ &= \frac{3}{s^2+1} + e^{-\pi s}\frac{3}{s^2+1} = \frac{3(1+e^{-\pi s})}{s^2+1}\end{aligned}$$

Then

$$X(s) = \frac{3(1+e^{-\pi s})}{(s^2+1)(s^2+4)} = (1+e^{-\pi s})\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] \qquad \text{(partial fractions)}$$

$$
\begin{aligned}
&= \mathcal{L}\left\{(\sin t - \frac{1}{2}\sin 2t) + u(t-\pi)(\sin(t-\pi) - \frac{1}{2}\sin 2(t-\pi))\right\} \\
\implies x(t) &= \sin t - \frac{1}{2}\sin 2t - u(t-\pi)(\sin t + \frac{1}{2}\sin 2t) \\
&= (1-u(t-\pi))\sin t - \frac{1}{2}(1+u(t-\pi))\sin 2t \\
&= \begin{cases} \sin t - \frac{1}{2}\sin 2t & 0 \le t < \pi \\ -\sin 2t & t \ge \pi \end{cases}
\end{aligned}
$$

We are not interested in the solution for $t < 0$ since this is unphysical and Laplace transforms do not notice negative time. Given the initial conditions, we could imagine the solution is zero for all $t < 0$. The solution curve is plotted below, with the force curve for comparison. Note that even though f is not differentiable at $t = 0$ and π, the solution $x(t)$ is.