3.5 Two dimensional systems and their vector fields: Summary & Examples

Each example is a linear system of ODE with a critical point at the origin $(0,0)$.

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax + by \\ \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy \end{cases} \quad \leftrightsquigarrow \quad \mathbf{x}' = A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Otherwise said, $\mathbf{x}(t) = \mathbf{0}$ (or $\big(x(t), y(t)\big) = (0,0)$) is a constant solution to the system.
The behavior of non-constant solution curves depends on the eigenvalues of the matrix A. To help study this material, each case comes with an example system: for each,

1. Verify the computation of the eigenvalues and eigenvectors.
2. Describe the critical point using the terms sink/source/center/spiral, etc., and think about the asymptotic/long-behavior of the solution curves.
3. Cover up the picture and try to draw the phase portrait based only on the eigenvalues and eigenvectors. Does your picture match that drawn by the computer?

Case 1: Distinct Positive Eigenvalues $\lambda_1 > \lambda_2 > 0$, Improper Nodal Source

Each eigenvalue has a corresponding eigenvector, and we have the general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

We analyse how these solutions behave for large positive and negative t.

Large positive t: Since $\lambda_1, \lambda_2 > 0$, every non-constant solution satisfies

$$\lim_{t\to\infty} |\mathbf{x}(t)| = \infty$$

Since solution curves move away from the critical point $(0,0)$ as time increases, we call the origin a *source*. Moreover, for large positive t

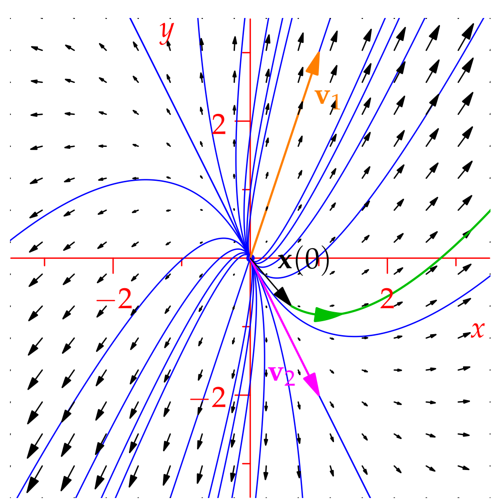
$$\lambda_1 > \lambda_2 \implies e^{\lambda_1 t} \gg e^{\lambda_2 t} \qquad (e^{10t} \text{ is much larger than } e^{5t})$$
$$\implies \mathbf{x}(t) \underset{t\to\infty}{\sim} c_1 e^{\lambda_1 t}\mathbf{v}_1$$

If $c_1 \neq 0$, then as $t \to \infty$ the solution vector $\mathbf{x}(t)$ *approaches being parallel* to $\mathbf{v}_1$.

Large negative t: For all solutions, $\lim_{t\to-\infty} |\mathbf{x}(t)| = 0$. Since for large negative t, we have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$ (both are very small), we conclude that,

$$\mathbf{x}(t) \underset{t\to-\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$$

All solution curves *depart the origin parallel to* $\mathbf{v}_2$.



The picture shows the phase portrait and direction field for the system $\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ 6 & 8 \end{pmatrix}\mathbf{x}$. It has eigenvalues $\lambda_1 = 10$ and $\lambda_2 = 5$, eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and general solution

$$\mathbf{x}(t) = c_1 e^{10t}\mathbf{v}_1 + c_2 e^{5t}\mathbf{v}_2$$

The arrow $\mathbf{x}(0) = \begin{pmatrix} 0.6 \\ -0.7 \end{pmatrix}$ is the position vector corresponding to the constants $c_1 = 0.1$, $c_2 = 0.5$, with the resulting trajectory evolving as t increases.

Case 2: Distinct Negative Eigenvalues $\lambda_1 < \lambda_2 < 0$, Improper Nodal Sink

The situation is the opposite of the case 1. The general solution remains $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$.

Large positive t: Since $\lambda_1, \lambda_2 < 0$, every solution satisfies $\lim_{t\to\infty} |\mathbf{x}(t)| = 0$. Since solution curves move towards the critical point $(0,0)$, we call the origin a *sink.* Moreover, for large positive t

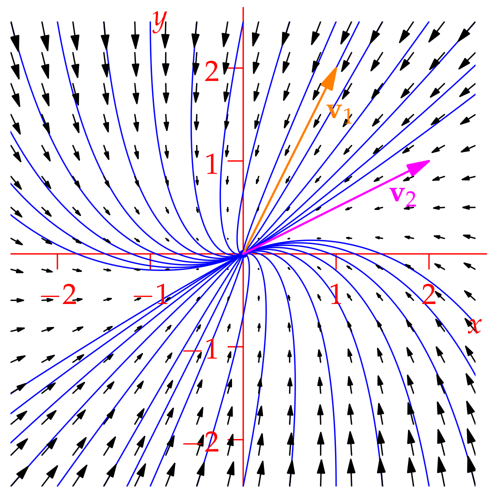
$$\lambda_1 < \lambda_2 \implies e^{\lambda_1 t} \ll e^{\lambda_2 t} \implies \mathbf{x}(t) \underset{t\to\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$$

The solutions $\mathbf{x}(t)$ *approach the origin parallel to* $\mathbf{v}_2$.

Large negative t: All non-zero solutions satisfy $\lim_{t\to-\infty} |\mathbf{x}(t)| = \infty$. Moreover,

$$\mathbf{x}(t) \underset{t\to-\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$$

All solution curves approach infinity *parallel to* $\mathbf{v}_2$.



The example shows the phase portrait and direction field of the system $\mathbf{x}' = \begin{pmatrix} -5 & -2 \\ 2 & -10 \end{pmatrix}\mathbf{x}$ with eigenvalues/vectors

$$\lambda_1 = -9 \quad \lambda_2 = -6 \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

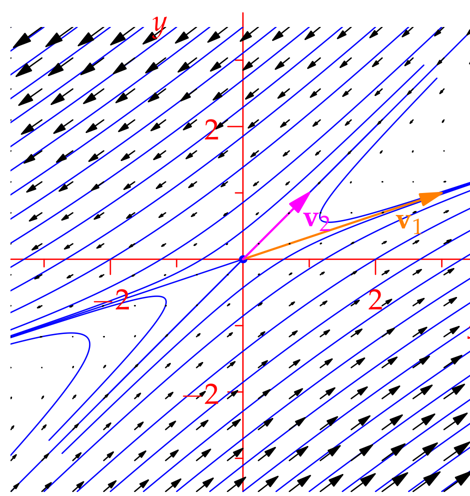
Case 3: Opposite Sign Eigenvalues $\lambda_1 > 0 > \lambda_2$, Saddle Point

We still have two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and general solution $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. However,

$$\lim_{t\to\infty} |\mathbf{x}(t)| = \begin{cases} 0 & \text{if } c_1 = 0 \\ \infty & \text{if } c_1 \neq 0 \end{cases} \qquad \text{and} \qquad \lim_{t\to-\infty} |\mathbf{x}(t)| = \begin{cases} 0 & \text{if } c_2 = 0 \\ \infty & \text{if } c_2 \neq 0 \end{cases}$$

Large positive t: we have $\mathbf{x}(t) \underset{t\to\infty}{\sim} c_1 e^{\lambda_1 t}\mathbf{v}_1$, so trajectories become almost parallel to $\mathbf{v}_1$.

Large positive t: we have $\mathbf{x}(t) \underset{t\to-\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$, so solutions are approximately parallel to $\mathbf{v}_2$.



The picture shows the system $\mathbf{x}' = \begin{pmatrix} 5 & -9 \\ 3 & -7 \end{pmatrix}\mathbf{x}$ with eigenvalues/vectors

$$\lambda_1 = 2 \quad \lambda_2 = -4 \quad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The picture shows why we call the critical point a *saddle*: the solution curves are reminiscent of the level curves near a saddle point in multivariable calculus.

Case 4: Repeated Non-zero Real Eigenvalues $\lambda_1 = \lambda_2 \neq 0$, Nodal Source/Sink

When a 2×2 matrix has a single eigenvalue λ, there are two possibilities:

(a) $A = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is a multiple of the identity matrix. *Any* non-zero vector $\mathbf{v}$ is an eigenvector and the general solution to $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = e^{\lambda t}\mathbf{v} = e^{\lambda t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

$\lambda > 0$: All non-zero trajectories move away from the origin in a straight line. We call $(0,0)$ a *proper nodal source.*

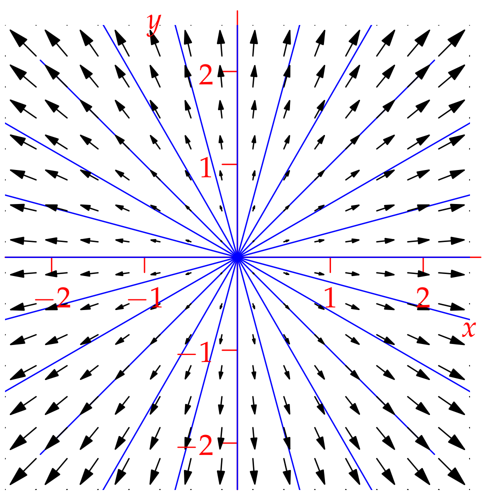
$\lambda < 0$: All non-zero trajectories move towards the origin in a straight line. We call $(0,0)$ a *proper nodal sink.*

(b) (Non-examinable) A has only one eigenvector $\mathbf{v}_1$: in this case the general solution is

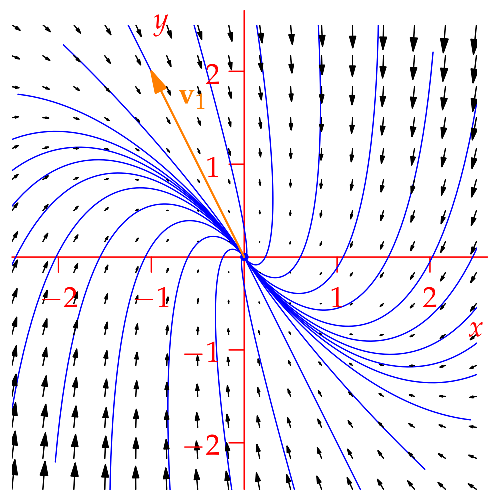
$$\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2(t\mathbf{v}_1 + \mathbf{v}_2))$$

where $\mathbf{v}_2$ is a *generalized eigenvector* of A. We have $\mathbf{x}(t) \underset{t\to\pm\infty}{\sim} c_2te^{\lambda t}\mathbf{v}_1$ *regardless* of the sign of λ. All trajectories are tangent to $\mathbf{v}_1$ at the origin *and* as $t \to \infty$. We call $(0,0)$ an *improper nodal source* (if $\lambda > 0$) or *sink* (if $\lambda < 0$).

The second picture below shows the solution curves for the system $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -4 & -6 \end{pmatrix}\mathbf{x}$, with repeated eigenvalues $\lambda_1 = \lambda_2 = -4$, eigenvector $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and generalized eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. It is hard to see from the picture, but the solution curves come from infinity almost parallel to the eigenvector, before reversing direction to approach the origin again parallel to the same direction.



Subcase (a): Proper Nodal Source



Subcase (b): Improper Nodal Sink

Case 5: Complex Eigenvalues $\lambda = p \pm iq$ $(q > 0)$, Spirals and Centers

If the eigenvector $\mathbf{v} = \mathbf{p} + i\mathbf{q}$ corresponds to λ, then $\overline{\mathbf{v}} = \mathbf{p} - i\mathbf{q}$ is the eigenvector corresponding to $\overline{\lambda}$. The real and imaginary parts of $e^{\lambda t}\mathbf{v}$ are independent solutions. Applying Euler's formula and some trigonometric identities, the general solution can be written in the form

$$\mathbf{x}(t) = Ce^{pt}\big(\sin(qt - \gamma)\mathbf{p} + \cos(qt - \gamma)\mathbf{q}\big) \qquad (*)$$

where C and γ are arbitrary constants. There are two key parts to this solution.

Rotational: Recall that the parametrized circle $\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ rotates counter-clockwise from $\mathbf{i}$ to $\mathbf{j}$. Analogously (since $q > 0$), the general solution $(*)$ rotates around the origin from $\mathbf{q}$ towards $\mathbf{p}$.

Exponential: The sign of p determines the long term behavior:

Spiral Source: if $p > 0$, then trajectories spiral away from the origin.

Center: if $p = 0$, then trajectories are closed curves (ellipses).

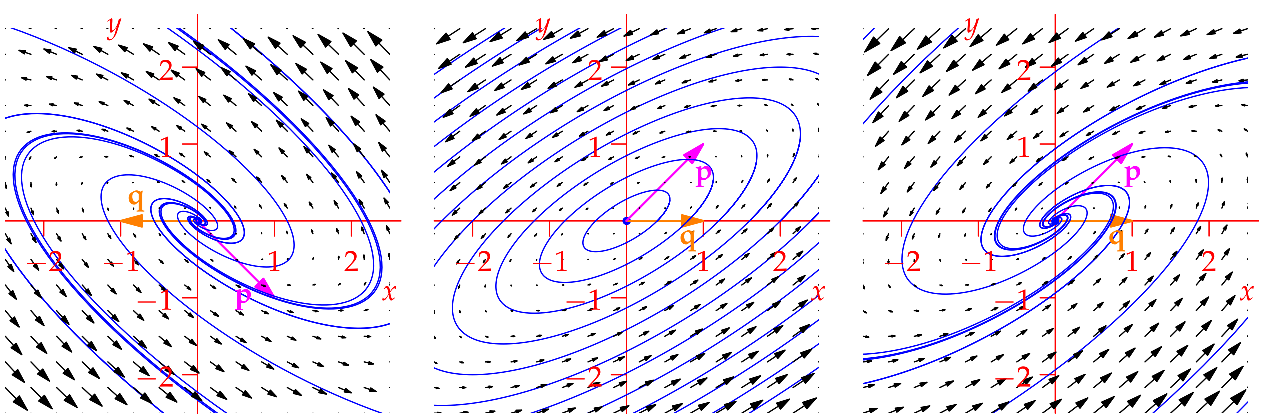
Spiral Sink: if $p < 0$, then trajectories spiral towards the origin.

An example of each is drawn below.

(a) $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix}\mathbf{x}$ has a spiral source: $\lambda = 1 \pm 2i$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

(b) $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\mathbf{x}$ has a center: $\lambda = \pm i$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

(c) $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 2 & -3 \end{pmatrix}\mathbf{x}$ has a spiral sink: $\lambda = -1 \pm 2i$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.



(a) Spiral Source (b) Center (c) Spiral Sink

It is often a waste of effort to compute the full solution $(*)$. To identify the type of critical point, first find the eigenvalues $\lambda = p \pm iq$: the sign of p says whether you have a source, sink, or center. Evaluating the direction vector at a single point tells you the direction of rotation.

For instance, in example (a), at the position $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the direction vector $\mathbf{x}' = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ points *upwards*: the spiral therefore rotates counter-clockwise!

Summary: Isolated Critical Points

All examples so far have dealt with the situation where the origin $(0, 0)$ is the only critical point of the system. The table summarizes how the possibilities depend on the eigenvalues.

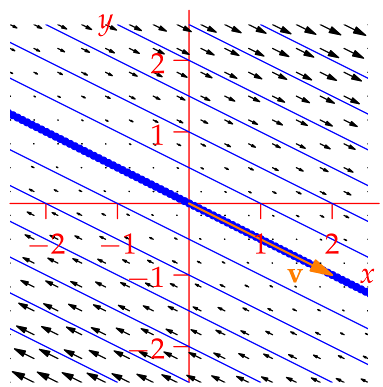
Case	Eigenvalues	Type of critical point
1	Both > 0, unequal	Improper nodal source
2	Both < 0, unequal	Improper nodal sink
3	Opposite Signs	Unstable saddle point
4	Both > 0, equal	Improper (rarely proper) nodal source
	Both < 0, equal	Improper (rarely proper) nodal sink
5	$p \pm iq$ with $p > 0$	Spiral source
	$p \pm iq$ with $p < 0$	Spiral sink
	$p \pm iq$ with $p = 0$	Stable center

Non-Isolated critical points (non-examinable)

A critical point of a system of ODEs is *isolated* if there are no other nearby critical points. Consider what it would mean $(0, 0)$ to be non-isolated the system $\mathbf{x}' = A\mathbf{x}$. We'd have to have another critical point (constant solution) $\mathbf{x}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. But this means $A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{0}$. The existence of such a point (linear algebra) is equivalent to $\det A = 0$, which is if and only if *zero is an eigenvalue of A*. This is why the summary table above covers all possibilities for when a 2×2 matrix has non-zero eigenvalues.
For the sake of completion, we briefly describe the three subcases for non-isolated critical points.

1. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Every point is critical. All solutions are constant points (*very boring!*).

2. $A = \begin{pmatrix} \alpha\beta & \pm\beta^2 \\ \mp\alpha^2 & -\alpha\beta \end{pmatrix} \neq 0$ has only $\lambda = 0$ and a single eigenvector $\mathbf{v} = \begin{pmatrix} \beta \\ \mp\alpha \end{pmatrix}$. The general solution is

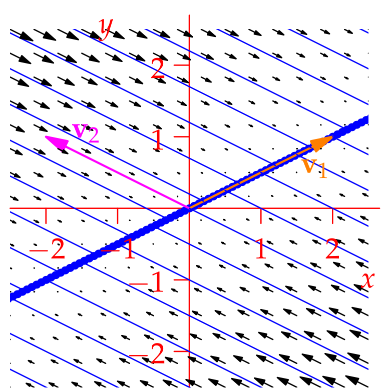
   

   $$\mathbf{x}(t) = c_1 \begin{pmatrix} \beta \\ \mp\alpha \end{pmatrix} + c_2 \left[\begin{pmatrix} \beta \\ \mp\alpha \end{pmatrix} t + \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \alpha \\ \pm\beta \end{pmatrix} \right]$$

   We have a line of critical points $k\mathbf{v}$, and all trajectories are parallel to $\mathbf{v}$.

   The picture shows the solutions to $\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \mathbf{x}$ with the single eigenvector $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

3. If $\lambda_1 = 0 \neq \lambda_2$, then the general solution is

   $$\mathbf{x}(t) = c_1 \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

   Since $A\mathbf{v}_1 = \mathbf{0}$, we have a line of critical points $k\mathbf{v}_1$. All other trajectories are straight lines parallel to $\mathbf{v}_2$; these move away ($\lambda_2 > 0$) or approach ($\lambda_2 < 0$) the line of critical points.

   The picture shows the solutions to $\mathbf{x}' = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{x}$ with $\lambda_1 = 0$, $\lambda_2 = -4$ and eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.