3.5 Two dimensional systems and their vector fields

Isolated critical points of linear systems

Definition. *A critical point of a system of ODEs is* isolated *if there are no other critical points nearby.*

Consider the following linear system with critical point $(0,0)$:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax + by \\ \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy \end{cases} \quad \leftrightsquigarrow \quad \mathbf{x}' = A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} \tag{$*$}$$

Does it have any other critical points?

Recall that the point (x_0, y_0) is critical if and only if is a constant solution to the differential equation: that is, if and only if it solves the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We conclude that $(*)$ has a *non-zero* critical point $(x_0, y_0) \neq (0,0)$ if and only if the vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an eigenvector of the matrix A with *eigenvalue* 0.

Alternatively: A has characteristic equation: $\lambda^2 - (a+d)\lambda + \underbrace{ad - bc}_{\det A} = 0$.

Plainly this has a root $\lambda = 0 \iff \det A = 0$.

We conclude: $(*)$ has an isolated critical point at the origin if and only if *both of its eigenvalues are non-zero*, and if and only if $\det A \neq 0$.

We now classify the types of isolated critical point dependent on the signs of the eigenvalues.

Each case has an example matrix: you should…

1. Calculate the eigenvalues and eigenvectors, making sure you get the answers given
2. Think about the asymptotic/long-behavior of the solution curves.
3. Try to construct the phase portrait yourself: see if it matches the picture given.
4. Associate the terms sink/source/center/spiral, etc. with each picture/eigenvalues
5. Generalized eigenvectors and non-isolated critical points are non-examinable; these pictures are only given for completion.
6. *Do not* memorize the vector form of the solution for complex eigenvalues: it is very messy and for interpretation purposes only!

Eigenvalues positive and distinct: $\lambda_1 > \lambda_2 > 0$

If the eigenvalues are distinct then they each have a corresponding eigenvector $\mathbf{v}_1, \mathbf{v}_2$ and we have the general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

Suppose that c_1, c_2 are *both non-zero.* Since $\lambda_1, \lambda_2 > 0$ it is immediate that

$$|\mathbf{x}(t)| \underset{t\to\infty}{\longrightarrow} \infty \qquad \text{and} \qquad |\mathbf{x}(t)| \underset{t\to-\infty}{\longrightarrow} 0$$

We can say more than this: *how* does the vector $\mathbf{x}(t)$ get larger/smaller?

For large positive t Note that[1] $e^{\lambda_1 t} \gg e^{\lambda_2 t}$. It follows that, as $t \to \infty$, the solution vector $\mathbf{x}(t)$ *approaches being parallel* to $\mathbf{v}_1$. We would write

$$\mathbf{x}(t) \underset{t\to+\infty}{\sim} c_1 e^{\lambda_1 t}\mathbf{v}_1$$

and say that $\mathbf{x}(t)$ is asymptotic to $c_1 e^{\lambda_1 t}\mathbf{v}_1$ as $t \to \infty$.

For large negative t We have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$ (both are very small). Since $e^{\lambda_1 t} \to 0$ *faster* than $e^{\lambda_2 t}$ we conclude that, as $t \to -\infty$, the solution vector $\mathbf{x}(t)$ *approaches being parallel* to $\mathbf{v}_2$. We would write

$$\mathbf{x}(t) \underset{t\to-\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$$

and say that $\mathbf{x}(t)$ is asymptotic to $c_2 e^{\lambda_2 t}\mathbf{v}_2$ as $t \to -\infty$.

To summarize Most[2] trajectories have leave the origin tangent to the eigenvector $\mathbf{v}_2$ and, as t increases, approach being parallel to the eigenvector $\mathbf{v}_1$. We call the critical point $(0,0)$ an *improper nodal source.*

The example shows the phase portrait and direction field of the system

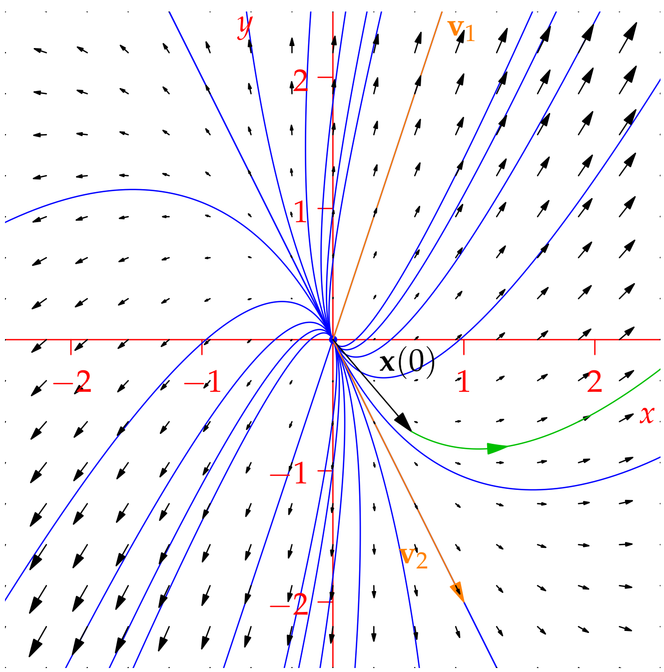
$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ 6 & 8 \end{pmatrix} \mathbf{x}$$

with eigenvalues/vectors

$$\lambda_1 = 10 \qquad \lambda_2 = 5$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The black arrow $\mathbf{x}(0) = \begin{pmatrix} 0.6 \\ -0.7 \end{pmatrix}$ is the position vector corresponding to the constants $c_1 = 0.1, c_2 = 0.5$. The green curve is the trajectory with this initial condition, evolving as t increases from zero.



[1] $e^{\lambda_1 t}$ is much much greater than $e^{\lambda_2 t}$, even though both will be exceptionally large!

[2] Those with initial conditions so that c_1, c_2 are *both* non-zero.

Eigenvalues negative and distinct: $\lambda_1 < \lambda_2 < 0$

The situation is exactly the opposite of the previous case. The general solution remains

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

where, if both c_1, c_2 are non-zero, then

$$|\mathbf{x}(t)| \underset{t\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad |\mathbf{x}(t)| \underset{t\to-\infty}{\longrightarrow} \infty$$

Asymptotically,

As $t \to \infty$ we have $e^{\lambda_1 t} \ll e^{\lambda_2 t}$, whence $\mathbf{x}(t) \underset{t\to+\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$

As $t \to -\infty$ we have $e^{\lambda_1 t} \gg e^{\lambda_2 t}$, whence $\mathbf{x}(t) \underset{t\to-\infty}{\sim} c_1 e^{\lambda_1 t}\mathbf{v}_1$

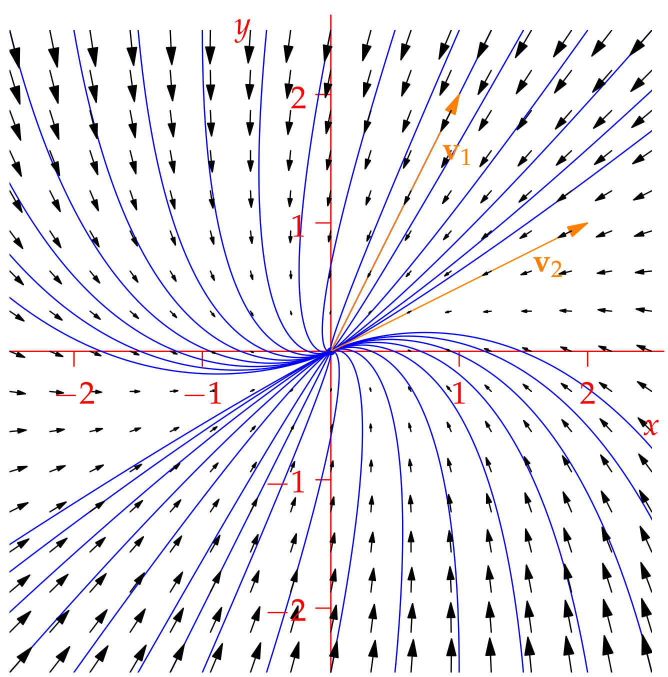
To summarize Most trajectories are nearly parallel to the eigenvector $\mathbf{v}_1$ when far from the origin ($t \to -\infty$) and approach the origin tangent to the eigenvector $\mathbf{v}_2$. We call the critical point $(0,0)$ an *improper nodal sink.*

The example shows the phase portrait and direction field of the system

$$\mathbf{x}' = \begin{pmatrix} -5 & -2 \\ 2 & -10 \end{pmatrix}\mathbf{x}$$

with eigenvalues/vectors

$$\lambda_1 = -9 \qquad \lambda_2 = -6 \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



Eigenvalues opposite signs: $\lambda_1 > 0 > \lambda_2$

We still have two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and general solution $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$, however we now have

$$\lim_{t\to\infty} |\mathbf{x}(t)| = \begin{cases} 0 & \text{if } c_1 = 0 \\ \infty & \text{if } c_1 \neq 0 \end{cases} \qquad \text{and} \qquad \lim_{t\to-\infty} |\mathbf{x}(t)| = \begin{cases} 0 & \text{if } c_2 = 0 \\ \infty & \text{if } c_2 \neq 0 \end{cases}$$

Asymptotically,

As $t \to \infty$ we have $e^{\lambda_1 t} \to \infty$ and $e^{\lambda_2 t} \to 0$, whence $\mathbf{x}(t) \underset{t\to+\infty}{\sim} c_1 e^{\lambda_1 t}\mathbf{v}_1$

As $t \to -\infty$ we have $e^{\lambda_1 t} \to 0$ and $e^{\lambda_2 t} \to \infty$, whence $\mathbf{x}(t) \underset{t\to-\infty}{\sim} c_2 e^{\lambda_2 t}\mathbf{v}_2$

For large negative t the tragectories are nearly parallel to the eigenvector $\mathbf{v}_2$, while for large positive t the trajectories are nearly parallel to $\mathbf{v}_1$.

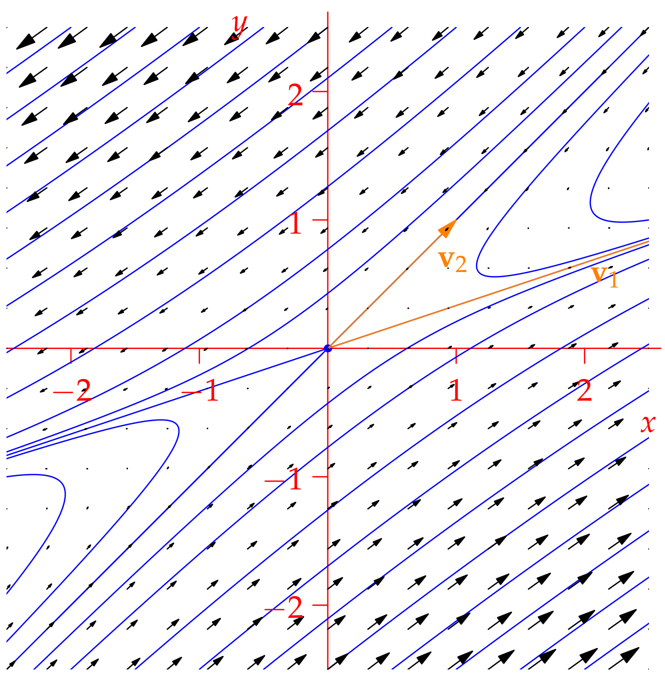
We call the critical point $(0,0)$ a *saddle point.*

The example shows the phase portrait and direction field of the system

$$\mathbf{x}' = \begin{pmatrix} 5 & -9 \\ 3 & -7 \end{pmatrix} \mathbf{x}$$

with eigenvalues/vectors

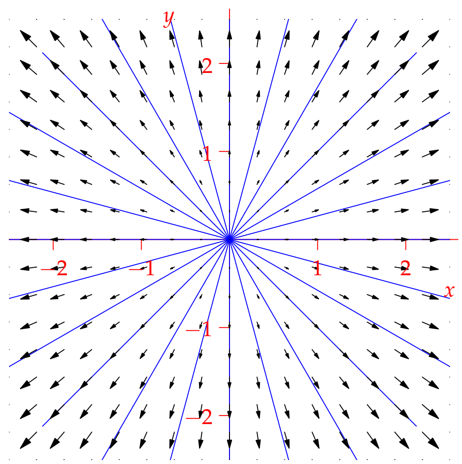
$$\lambda_1 = 2 \qquad \lambda_2 = -4 \qquad \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



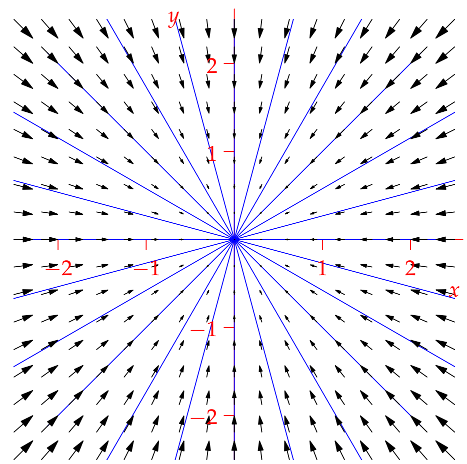
Repeated real eigenvalues: $\lambda_1 = \lambda_2 \neq 0$

When a 2×2 matrix has a single eigenvalue λ, there are two possibilities:

1. $A = \lambda I = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ is a multiple of the identity matrix. Then *any* non-zero vector $\mathbf{v}$ is an eigenvector and so the general solution is $\mathbf{x}(t) = e^{\lambda t}\mathbf{v} = e^{\lambda t}\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$. All non-zero trajectories move away from ($\lambda > 0$) or towards ($\lambda < 0$) the origin in a straight line. We call $(0,0)$ a *proper nodal source/sink* depending on the sign of λ.



A proper nodal source

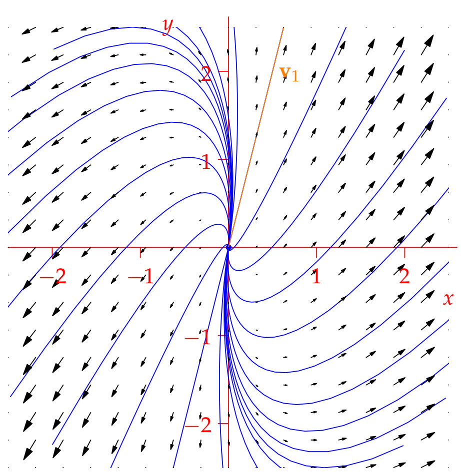


A proper nodal sink

2. A has only one eigenvector $\mathbf{v}_1$: in this case the general solution is

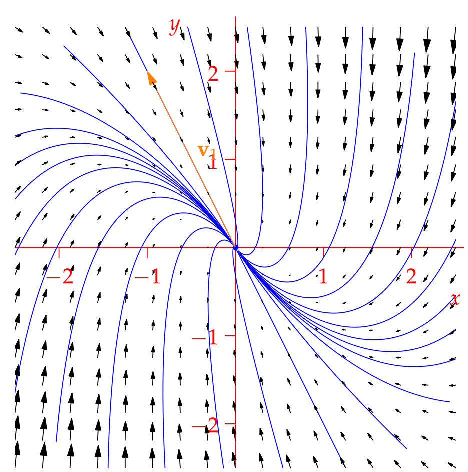
$$\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2(t\mathbf{v}_1 + \mathbf{v}_2))$$

where $\mathbf{v}_2$ is a *generalized eigenvector*[3] of A. The asymptotic behavior is $\mathbf{x}(t) \underset{t\to\pm\infty}{\sim} c_2 t e^{\lambda t}\mathbf{v}_1$ *regardless* of the sign of λ. All trajectories become tangent to $\mathbf{v}_1$ at the origin *and* as $t \to \infty$.
$(0,0)$ is an *improper nodal source* (if $\lambda > 0$) or *sink* (if $\lambda < 0$).



$$\mathbf{x}' = \begin{pmatrix} 15 & -1 \\ 16 & 7 \end{pmatrix}\mathbf{x} \qquad \lambda_1 = \lambda_2 = 11$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -4 & -6 \end{pmatrix}\mathbf{x} \qquad \lambda_1 = \lambda_2 = -4$$

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

[3]This is non-examinable. Generalized eigenvectors will be covered in a later linear algebra course.

Complex eigenvalues: $\lambda = p + iq$, $\overline{\lambda} = p - iq \quad (q \neq 0)$

If the eigenvector $\mathbf{v} = \mathbf{p} + i\mathbf{q}$ correspoinds to λ, then $\overline{\mathbf{v}} = \mathbf{p} - i\mathbf{q}$ is the eignevector of$\overline{\lambda}$.
The general solution is $\mathbf{x}(t) = c_1 \Re(e^{\lambda t}\mathbf{v}) + c_2 \Im(e^{\lambda t}\mathbf{v})$. Applying Euler's formula and some trigonometric identities we may write the general solution as

$$\mathbf{x}(t) = Ce^{pt}\big(\sin(qt - \gamma)\mathbf{p} + \cos(qt - \gamma)\mathbf{q}\big) \qquad (*)$$

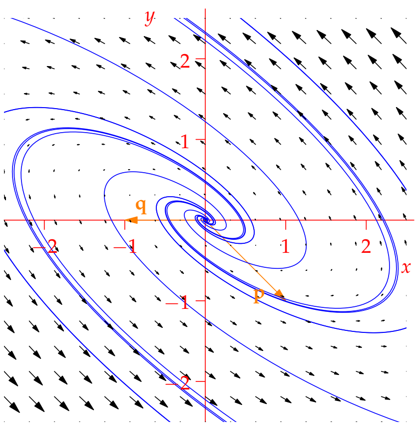
where C and γ are arbitrary constants.
Consider the trigonometric part of the solution. Recalling that the parameterized circle $\cos t\mathbf{i} + \sin t\mathbf{j}$ rotates from $\mathbf{i}$ to $\mathbf{j}$, we see that that (if $q > 0$) then the general solution $(*)$ rotates around the origin from $\mathbf{q}$ towards $\mathbf{p}$.
The exponential part of the solution is easier to consider:

$p > 0$ Trajectories spiral outwards: $(0,0)$ is a *spiral source*

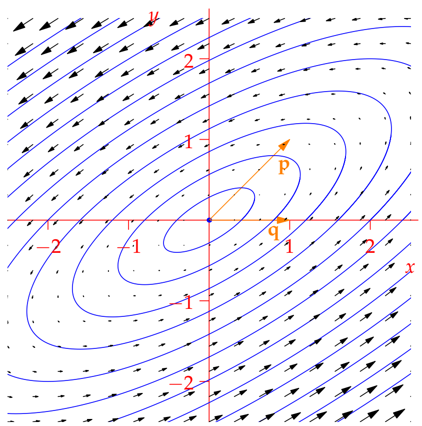
$p = 0$ Trajectories are closed ellipses: $(0,0)$ is a *stable center*

$p < 0$ Trajectories spiral inwards: $(0,0)$ is a *spiral sink*

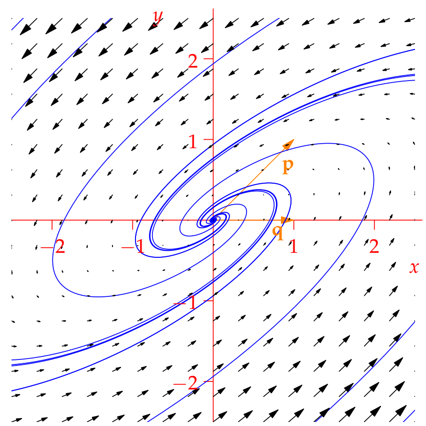
An example of each is drawn below.



$$A = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix}$$
$$\lambda = 1 + 2i$$
$$\mathbf{v} = \begin{pmatrix} 1 - i \\ -1 \end{pmatrix}$$



$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$
$$\lambda = i$$
$$\mathbf{v} = \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}$$
$$\lambda = -1 + 2i$$
$$\mathbf{v} = \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}$$

In practice it is wasted effort to write the solution in the form $(*)$. Simply compute the eigenvalues $\lambda = p \pm iq$ and observe the sign of p: this gives you a spiral source or sink, or a center. To determine the direction of rotation, choose a point (say (1,0)) and compute the direction vector at that point. With our first example, if $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\mathbf{x}' = A\mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ points *upwards*. Therefore the spiral rotates counterclockwise!

Summary: isolated critical points of linear systems

Make sure you are comfortable with this table and that you know *why* is line is true

Eigenvalues	Type of critical point
Both > 0, unequal	Improper nodal source
Both < 0, unequal	Improper nodal sink
Opposite Signs	Unstable saddle point
Both > 0, equal	Improper (rarely proper) nodal source
Both < 0, equal	Improper (rarely proper) nodal sink
$p \pm iq$ with $p > 0$	Spiral source
$p \pm iq$ with $p < 0$	Spiral sink
$p \pm iq$ with $p = 0$	Stable center

Non-isolated critical points (non-examinable): if one or both eigenvales are zero

Both eigenvalues zero There are two cases:

1. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: every point is critical, all solutions are constant points (*very boring!*)
2. $A = \begin{pmatrix} \alpha\beta & \pm\beta^2 \\ \mp\alpha^2 & -\alpha\beta \end{pmatrix} \neq 0$ with a single eigenvector $\mathbf{v} = \begin{pmatrix} \beta \\ \mp\alpha \end{pmatrix}$. In this case the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \beta \\ \mp\alpha \end{pmatrix} + c_2 \left[\begin{pmatrix} \beta \\ \mp\alpha \end{pmatrix} t + \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \alpha \\ \pm\beta \end{pmatrix} \right]$$

We have a line of critical points $k\mathbf{v}$, and all trajectories are parallel to $\mathbf{v}$

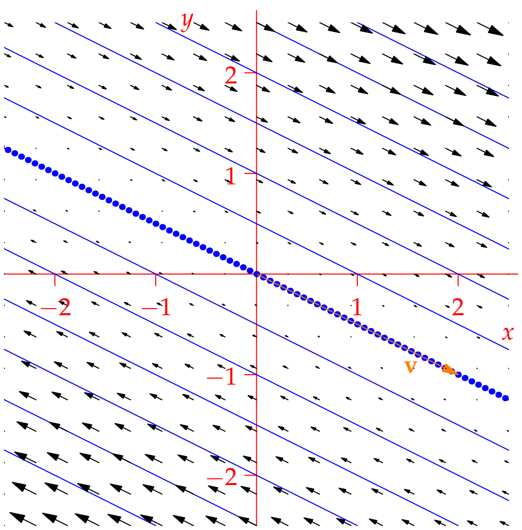
One eigenvalue zero $\lambda_1 = 0 \neq \lambda_2$

With two distinct eigenvalues, we have distinct eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and thus the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

Since $A\mathbf{v}_1 = \mathbf{0}$, we have a line of critical points $k\mathbf{v}_1$.
All other trajectories are straight lines parallel to $\mathbf{v}_2$
These move away ($\lambda_2 > 0$) or approach ($\lambda_2 < 0$) the line of critical points

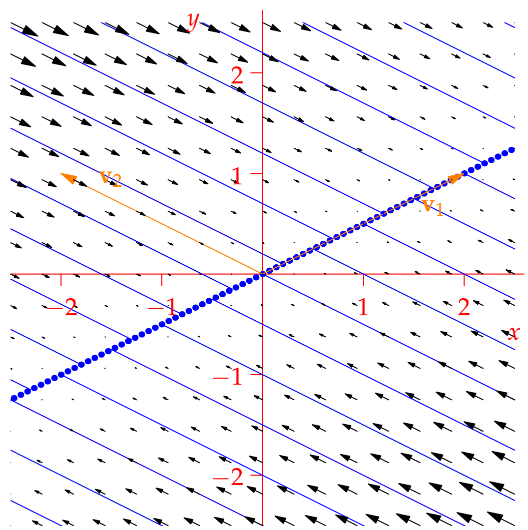
Pictures for the two non-trivial cases are below.



$$\mathbf{x}' = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \mathbf{x}$$

Both eigenvalues zero

One eigenvector $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



$$\mathbf{x}' = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$$

One eigenvalue zero, $\lambda_2 = -4$

$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$