2.5 Nonhomogeneous Equations

Example. Solve the initial value problem

$$\begin{cases} y''' - 3y'' + 3y' - y = 2e^x \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -1 \end{cases}$$

Solution. First find the complementary function. The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \iff (\lambda - 1)^3 = 0 \iff \lambda = 1.$$

We thus have a repeated root. The complementary function is therefore

$$y_{\mathrm{CF}}(x) = (c_1 + c_2 x + c_3 x^2)e^x.$$

Now search for a particular integral. Our first guess might be $y_{\mathrm{PI}}(x) = ae^x$, where a is unknown. However this solves the homogeneous equation, so we are forced to multiply by x. Indeed xe^x and x^2e^x also solve the homogeneous equation, so our guess should therefore be

$$y_{\mathrm{PI}}(x) = ax^3e^x.$$

Now differentiate and substitute into the ODE:

$$\begin{aligned}
y'_{\mathrm{PI}}(x) &= ae^x(x^3 + 3x^2), \\
y''_{\mathrm{PI}}(x) &= ae^x(x^3 + 3x^2 + 3x^2 + 6x) = ae^x(x^3 + 6x^2 + 6x), \\
y'''_{\mathrm{PI}}(x) &= ae^x(x^3 + 6x^2 + 6x + 3x^2 + 12x + 6) = ae^x(x^3 + 9x^2 + 18x + 6), \\
y'''_{\mathrm{PI}} - 3y''_{\mathrm{PI}} + 3y'_{\mathrm{PI}} - y_{\mathrm{PI}} &= ae^x(x^3 + 9x^2 + 18x + 6 - 3(x^3 + 6x^2 + 6x) + 3(x^3 + 3x^2) - x^3) \\
&= ae^x\left[(1 - 3 + 3 - 1)x^3 + (9 - 18 + 9)x^2 + (18 - 18)x + 6\right] \\
&= 6ae^x \\
&= 2e^x \iff a = \frac{1}{3}.
\end{aligned}$$

The general solution is therefore

$$y(x) = y_{\mathrm{CF}}(x) + y_{\mathrm{PI}}(x) = \left(c_1 + c_2 x + c_3 x^2 + \frac{1}{3}x^3\right)e^x.$$

To find the constants, differentiate and apply the initial conditions:

$$y(0) = 1 \implies c_1 = 1.$$

$$\begin{aligned}
y'(x) &= \left(1 + c_2 + (c_2 + 2c_3)x + (c_3 + 1)x^2 + \frac{1}{3}x^3\right)e^x \\
y'(0) &= 0 \implies 1 + c_2 = 0 \implies c_2 = -1.
\end{aligned}$$

$$y''(x) = \left(-1 + 2c_3 + (1 + 4c_3)x + (c_3 + 2)x^2 + \frac{1}{3}x^3\right) e^x$$
$$y''(0) = -1 \Longrightarrow -1 + 2c_3 = -1 \Longrightarrow c_3 = 0.$$

The particular solution satisfying the initial conditions is therefore

$$y(x) = \left(1 - x + \frac{1}{3}x^3\right) e^x.$$