

An Awkward Non-homogeneous Initial Value Problem (Section 2.5)

Solve the initial value problem

$$\begin{cases} y''' - 3y'' + 3y' - y = 2e^x \\ y(0) = 1, \quad y'(0) = 0 \quad y''(0) = -1 \end{cases}$$

Solution First find the complementary function. The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \iff (\lambda - 1)^3 = 0 \iff \lambda = 1$$

We thus have a repeated root. The complementary function is

$$y_C(x) = (c_1 + c_2x + c_3x^2)e^x$$

Now search for a particular integral. Our first guess might be $y_P(x) = ae^x$, where a is unknown. However this solves the homogeneous equation! Following the standard mantra, we keep multiplying by x : since xe^x and x^2e^x also solve the homogeneous equation, our guess should therefore be

$$y_P(x) = ax^3e^x$$

Now differentiate and substitute into the ODE:

$$y'_P(x) = ae^x(x^3 + 3x^2)$$

$$y''_P(x) = ae^x(x^3 + 3x^2 + 3x^2 + 6x) = ae^x(x^3 + 6x^2 + 6x)$$

$$y'''_P(x) = ae^x(x^3 + 6x^2 + 6x + 3x^2 + 12x + 6) = ae^x(x^3 + 9x^2 + 18x + 6)$$

$$\begin{aligned} y''' - 3y'' + 3y' - y_P &= ae^x(x^3 + 9x^2 + 18x + 6 - 3(x^3 + 6x^2 + 6x) + 3(x^3 + 3x^2) - x^3) \\ &= ae^x[(1 - 3 + 3 - 1)x^3 + (9 - 18 + 9)x^2 + (18 - 18)x + 6] \\ &= 6ae^x \\ &= 2e^x \iff a = \frac{1}{3} \end{aligned}$$

The general solution is therefore

$$y(x) = y_C(x) + y_P(x) = \left(c_1 + c_2x + c_3x^2 + \frac{1}{3}x^3 \right) e^x$$

To find the constants, differentiate and apply the initial conditions:

$$y(0) = 1 \implies c_1 = 1$$

$$y'(x) = \left(1 + c_2 + (c_2 + 2c_3)x + (c_3 + 1)x^2 + \frac{1}{3}x^3 \right) e^x$$

$$y'(0) = 0 \implies 1 + c_2 = 0 \implies c_2 = -1$$

$$y''(x) = \left(-1 + 2c_3 + (1 + 4c_3)x + (c_3 + 2)x^2 + \frac{1}{3}x^3 \right) e^x$$

$$y''(0) = -1 \implies -1 + 2c_3 = -1 \implies c_3 = 0$$

The particular solution satisfying the initial conditions is therefore

$$y(x) = \left(1 - x + \frac{1}{3}x^3 \right) e^x$$