Population Models Examples (1.6 Autonomous Equations)

Basic Assumption: the population dynamics of a group depend on two functions of time:

Birth Rate: $\beta(t, P)$ = average number of births per group member, per unit time

Death Rate: $\delta(t, P)$ = average number of deaths per group member, per unit time

Let $P(t)$ = population at time t. Between t and $t + \mathrm{d}t$, the total births are approximately $\beta P\,\mathrm{d}t$, and the total deaths $\delta P\,\mathrm{d}t$. The net change in population is their difference:

$$\mathrm{d}P = (\beta - \delta)P\,\mathrm{d}t \implies \frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$$

Different models depend on choices, observations and predictions of birth and death rates. One enormous problem with this approach is that calculus is *continuous*, whereas many populations are *discrete.* These models have no problem reporting, say, $P(3) = 12.237$ giraffes! Any real-world application requires *interpretation* and an acknowledgement that models only ever provide *approximations* of reality.

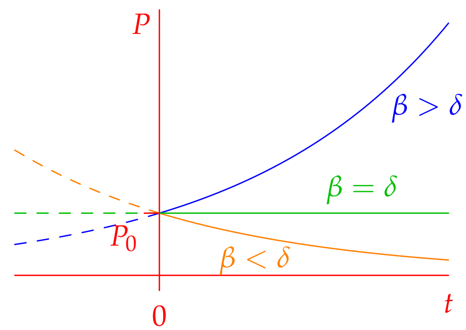
In what follows we discuss three commonly encountered models.

1. Natural Growth Model

Suppose both β and δ are constant. If we write $k = \beta - \delta$, then this is easily solved:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \implies P(t) = P_0 e^{kt}$$

where P_0 is the initial population (at time $t = 0$).
If $k < 0$ (death rate greater than birth rate), this is usually called a natural decay model.

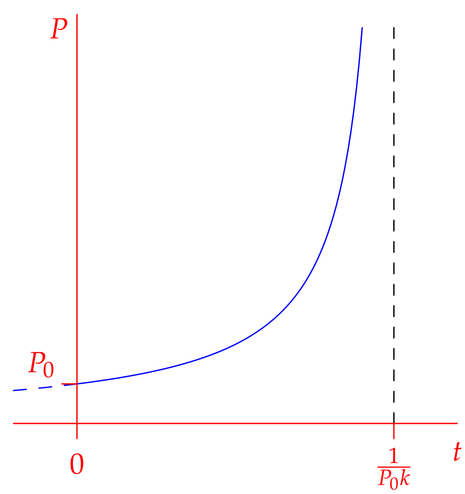


2. Growth rate proportional to population

Let $\beta - \delta = kP$ with $k > 0$ constant. Sometimes called a "dumb breeding" model, this might describe a situation where there is no death ($\delta = 0$) and new members of the population are created whenever two existing members bump into each other. The general solution is easily found using separation of variables:

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P = kP^2 &\implies \int_{P_0}^{P} \tilde{P}^{-2}\,\mathrm{d}\tilde{P} = \int_0^t k\,\mathrm{d}\tilde{t} \\ &\implies -P^{-1} + P_0^{-1} = kt \\ &\implies P(t) = \frac{P_0}{1 - P_0 kt}\end{aligned}$$

The population blows up in finite time: $\lim_{t \to \frac{1}{P_0 k}} P(t) = \infty$.



This model is unlikely to be accurate for long-term modeling of some animal population, though it might be more appropriate for modeling, say, an explosion (e.g. number of neutrons created by an exploding atom-bomb).

3. The Logistic equation

In the logistic model, the growth rate $\beta - \delta$ is assumed to be a *linear* function of P:

$$\beta - \delta = k\big(M - P(t)\big)$$

where M and k are constant. Supposing M and k are positive,

- $P(t) > M \implies \beta - \delta < 0 \implies P(t)$ *decreases* towards M.
- $P(t) < M \implies \beta - \delta > 0 \implies P(t)$ *increases* towards M.

The resulting ODE is the *Logistic Differential Equation*:

$$\frac{dP}{dt} = kP(M - P)$$

This model can arise in many situations. For instance:

Limited environment Environments typically have space or food constraints (e.g., fish in a pond). As a population grows, the growth rate $\beta - \delta$ tends to decrease: less food/space means fewer offspring and/or more deaths. In this case M measures the available space or *carrying capacity* of the environment, and $\frac{1}{k}(\beta - \delta) = M - P$ measures the available room for expansion.

Disease/Rumor spread The rate of spread of an infectious disease might be modelled logistically: a high rate of spread ($\frac{dP}{dt}$) arises when both the infected population P and the un-exposed population $M - P$ are large.

Remember that "population" is a very general term, it doesn't have to mean "of animals." $P(t)$ might represent a number of bacteria, of atoms, a mass, or a stock price! Depending on context, negative values for $P(t)$ might even make sense.

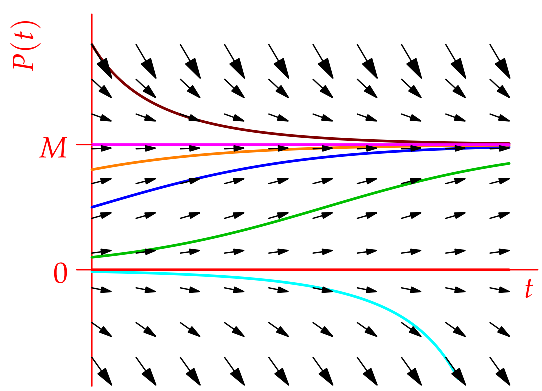
The general solution to the logistic equation can be explicitly computed via separation of variables:

$$\int_0^t k\,d\tilde{t} = \int_{P_0}^P \frac{1}{\tilde{P}(M-\tilde{P})}\,d\tilde{P} = \frac{1}{M}\int_{P_0}^P \frac{1}{\tilde{P}} + \frac{1}{M-\tilde{P}}\,d\tilde{P} \qquad \text{(partial fractions)}$$

$$\implies Mkt = \ln\frac{(M-P_0)P}{P_0(M-P)}$$

$$\implies \cdots$$

$$\implies P(t) = \frac{P_0 M}{P_0 + (M-P_0)e^{-Mkt}}$$



The slope field and phase portrait are drawn when k and M are positive. In such a situation, $P(t) \equiv M$ is a stable equilibrium solution: if $P_0 > 0$, then $\lim_{t\to\infty} P(t) = M$. By contrast, $P(t) \equiv 0$ is unstable.

Fish Pond Examples Here are several examples based on a population of fish in a pond. Other applications are in the exercises. Throughout, we assume that the population $P(t)$ has a constant birth rate $\beta = 2$ fish, per fish, per year, and a linearly increasing death rate

$$\delta(t) = 1 + 0.01P \quad \text{per fish, per year}$$

1. If the pond is initially seeded with $P_0 = 25$ fish, find the population of fish after t years.

   Solution: Using the given expressions for the birth and death rates, the required model is

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P = (1 - 0.01P)P = 0.01P(100 - P)$$

   whence $k = 0.01$ and $M = 100$ in the logistic model. The number of fish after t years is

   $$P(t) = \frac{P_0M}{P_0 + (M - P_0)e^{-Mkt}} = \frac{2500}{25 + 75e^{-t}} = \frac{100}{1 + 3e^{-t}}$$

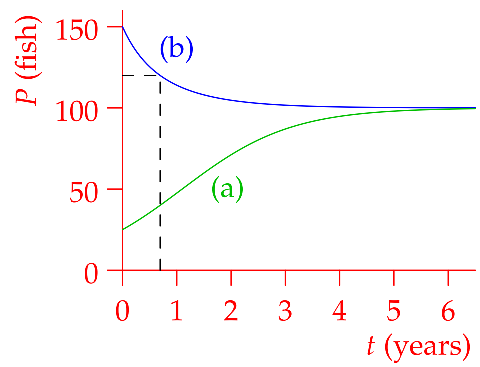
2. Suppose instead that $P_0 = 150$ fish are introduced to the pond. How long does it take for the population to reach 120 fish?

   Solution: We have the same model, but a different initial condition $P_0 = 150$. This time

   $$P(t) = \frac{P_0M}{P_0 + (M - P_0)e^{-Mkt}} = \frac{300}{3 - e^{-t}}$$

   The population reaches 120 fish when

   $$120 = \frac{300}{3 - e^{-t}} \iff 3 - e^{-t} = \frac{5}{2}$$
   $$\iff e^{-t} = \frac{1}{2}$$
   $$\iff t = \ln 2 \text{ years} \approx 8.3 \text{ months}$$

   In both cases the fish population approaches the stable equilibrium $M = 100$.

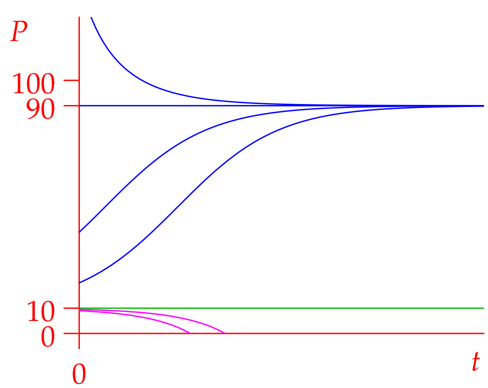
3. (Harvesting) Suppose $h = 9$ additional fish are removed per unit time ($h\,\mathrm{d}t$ fish in an infinitesimal time interval $\mathrm{d}t$). Determine how the fish population behaves.

   Solution: The result is a new differential equation:

   $$\mathrm{d}P = kP(M - P)\,\mathrm{d}t - h\,\mathrm{d}t \implies \frac{\mathrm{d}P}{\mathrm{d}t} = kP(M - P) - h = \frac{P}{100}(100 - P) - 9$$

   Using the quadratic formula or by factorizing, we obtain

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{1}{100}(P^2 - 100P + 900)$$
   $$= -\frac{1}{100}(P - 90)(P - 10)$$

   By considering how the sign of $\frac{\mathrm{d}P}{\mathrm{d}t}$ depends on P, we see that $P_1 = 90$ is a stable equilibrium, whereas $P_2 = 10$ is unstable. If the initial population $P(0) > 10$, then the fish eventually stabilize at the new *lower* equilibrium $P_1 = 90 < M = 100$; if $P(0) < 10$, the fish die out.