1.4 Linear First-order Equations

Example. A reservoir holds 160,000 acre ft of water, 0.25% of which is a pollutant. Each day, 10,000 acre ft of water with pollutant concentration 0.05% enters the reservoir, and the same volume of well-mixed water leaves.

(a) Set up a differential equation for the volume $V(t)$ of pollutant in the reservoir after t days.

(b) How long does it take for the pollutatant level to fall to 0.1%?

Solution. (a) Set up notation. Over a time interval Δt, we have an increase in pollutant volume ΔV. It should be clear that

$$\Delta V \approx (V_{\text{in}} - V_{\text{out}})\Delta t$$

where V_{in} and V_{out} are the volumes of pollutant entering and leaving the reservoir *each day.* Now,

$$\begin{aligned} V_{\text{in}} &= (\text{Volume of water entering each day}) \cdot (\text{Concentration}) = 10,000 \cdot \frac{0.05}{100} = 5 \text{ acre ft}, \\ V_{\text{out}} &= (\text{Fraction of reservoir leaving each day}) \cdot (\text{Volume of pollutant in reservoir}) \\ &= 10,000 \cdot \frac{V(t)}{160,000} = \frac{1}{16} V \text{ acre ft.} \end{aligned}$$

Therefore

$$\frac{\Delta V}{\Delta t} \approx 5 - \frac{1}{16} V.$$

Taking the limit as $\Delta t \to 0$ yields the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 5 - \frac{1}{16} V.$$

(b) The equation is 1st order linear: $V' + \frac{1}{16} V = 5$, so we use the integrating factor method:[1]

$$r(t) = \exp\left(\int \frac{1}{16}\,\mathrm{d}t\right) = e^{t/16},$$

whence[2]

$$\begin{aligned} V(t) &= \frac{1}{r(t)}\left[\int 5r(t)\,\mathrm{d}t + A\right] = e^{-t/16}\left[\int 5e^{t/16}\,\mathrm{d}t + A\right] \\ &= 80 + Ae^{-t/16} \quad \text{acre ft.} \end{aligned}$$

Now consider the initial condition: if $C(t)$ is the concentration, then

$$C(0) = 0.25\% \Rightarrow V(0) = 160,000 \cdot \frac{0.25}{100} = 400 \Rightarrow A = 320.$$

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[1] Note the variables: V and t, not y and x!

[2] Using the arbitrary constant A to avoid confusion with *concentration.*

Thus

$$V(t) = 80(1 + 4e^{-t/16}) \text{ acre ft.}$$

$C(t) = 0.1\%$ when $V(t) = 160$, which is when

$$4e^{-t/16} = 1 \Rightarrow t = 16 \ln 4 \approx 22.18 \text{ days.}$$

Note: We could instead have gone for a differential equation for $C(t)$ directly

$$\frac{dC}{dt} + \frac{1}{16}C = \frac{1}{32,000}$$

but concentration, being unitless, is a little harder to work with.