The Reservoir Problem (1.4 Linear First-order Equations)

Example A reservoir holds 160,000 acre-ft of water, 0.25% of which is a pollutant. Each day, 10,000 acre-ft with pollutant concentration 0.05% enters the reservoir and the same volume of well-mixed water leaves.

1. Set up a differential equation for the volume $V(t)$ of pollutant in the reservoir after t days.
2. How long does it take for the pollutant level to fall to 0.1%?

Solution

1. Start by defining notation. Over an infinitesimal interval $\mathrm{d}t$, the pollutant volume increases by

$$\mathrm{d}V = V_{\text{in}} - V_{\text{out}}$$

where V_{in} and V_{out} are the volumes of pollutant entering and leaving the reservoir over the time interval Δt. Since $\mathrm{d}t$ is infinitesimal, we may compute

$$\begin{aligned} V_{\text{in}} &= (\text{Vol water entering each day}) \cdot (\text{Concentration}) \cdot (\text{Time interval}) \\ &= 10,000 \cdot \frac{0.05}{100} \cdot \Delta t = 5\,\mathrm{d}t \text{ acre ft} \\ V_{\text{out}} &= (\text{Vol water leaving each day}) \cdot (\text{Concentration}) \cdot (\text{Time interval}) \\ &= 10,000 \cdot \frac{V(t)}{160,000} \cdot \mathrm{d}t = \frac{1}{16} V\,\mathrm{d}t \text{ acre ft} \end{aligned}$$

(Since $\mathrm{d}t$ is infinitesimal, the concentration of pollutant in the reservoir from time t to $t + \mathrm{d}t$ is treated as a constant, namely $\frac{V(t)}{160,000}$.) We conclude that[1]

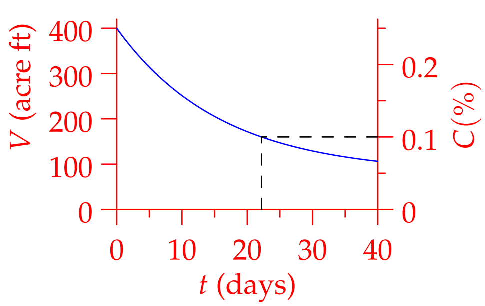
$$\mathrm{d}V = \left(5 - \frac{V}{16}\right) \mathrm{d}t \implies \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{16} = 5$$

2. The ODE is 1^{st}-order linear so we use the integrating factor method ($r(t) = \exp\left(\int \frac{1}{16} \mathrm{d}t\right) = e^{t/16}$):

$$\begin{aligned} V(t) &= \frac{1}{r(t)} \left[\int 5r(t)\,\mathrm{d}t + A \right] = e^{-t/16} \left[\int 5e^{t/16}\,\mathrm{d}t + A \right] \qquad (A \text{ is an arbitrary constant}) \\ &= 80 + Ae^{-t/16} \quad \text{acre-ft} \end{aligned}$$

where A is a constant. Now consider the initial condition: if $C(t)$ is the concentration, then

$$\begin{aligned} C(0) = 0.25\% &\implies V(0) = 160,000 \cdot \frac{0.25}{100} = 400 \\ &\implies A = 320 \\ &\implies V(t) = 80(1 + 4e^{-t/16}) \text{ acre-ft} \end{aligned}$$



$C(t) = 0.1\%$ when $V(t) = 160$, which is when

$$4e^{-t/16} = 1 \implies t = 16 \ln 4 \approx 22.18 \text{ days}$$

[1] We could instead have found a differential equation for $C(t)$ directly ($\frac{\mathrm{d}C}{\mathrm{d}t} + \frac{1}{16}C = \frac{1}{32,000}$), but concentration, being unitless, is less intuitive to work with.