1.2 Slope Fields: Examples and Euler's Method

General approach If we cannot easily solve a 1^{st}-order equation

$$\frac{dy}{dx} = f(x, y)$$

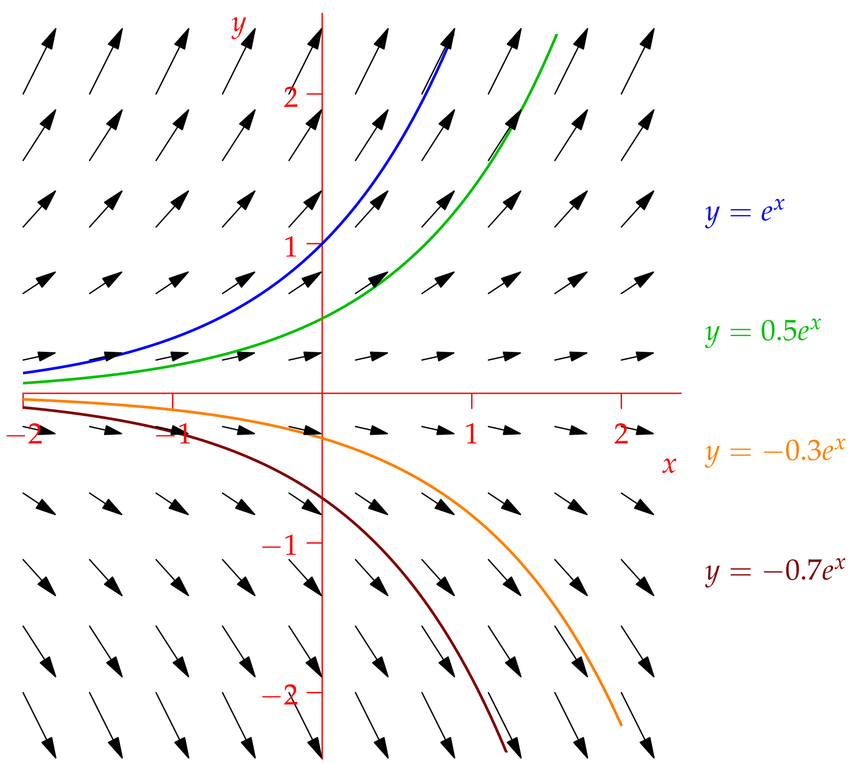
we can at least sketch the slope field

$$\mathbf{v}(x, y) = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix} = \mathbf{i} + f(x, y)\mathbf{j}$$

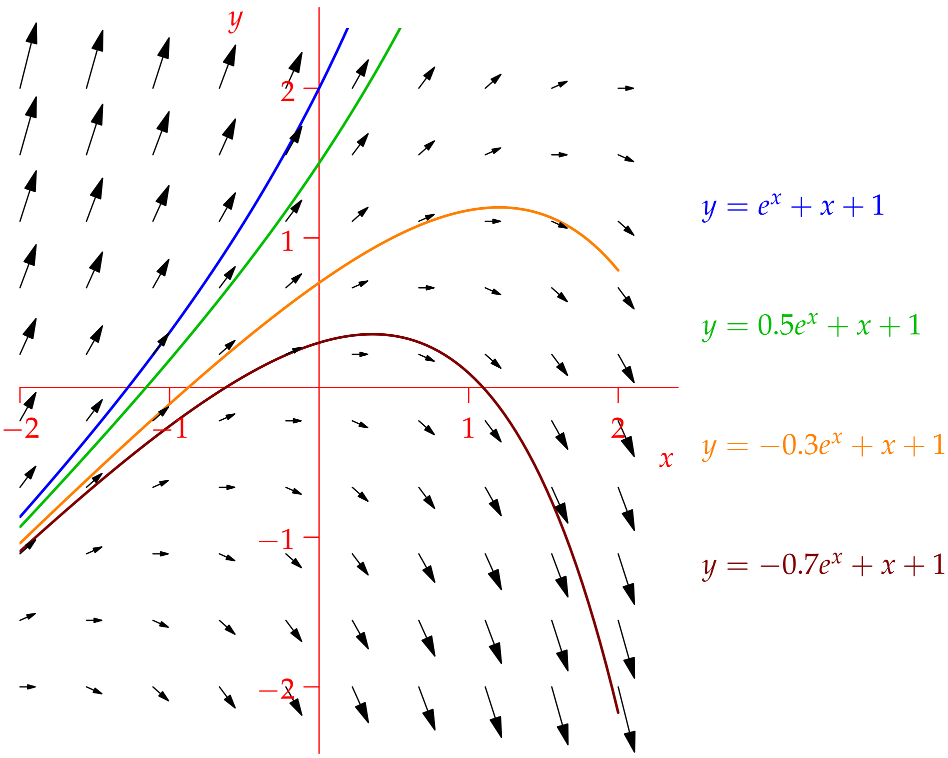
and guess/fit solution curves tangent to the arrows.

For the examples, exact solutions require separation of variables or the integrating factor method (sections 1.3 or 1.4). In general, it is unlikely that exact closed-form solutions can be found.

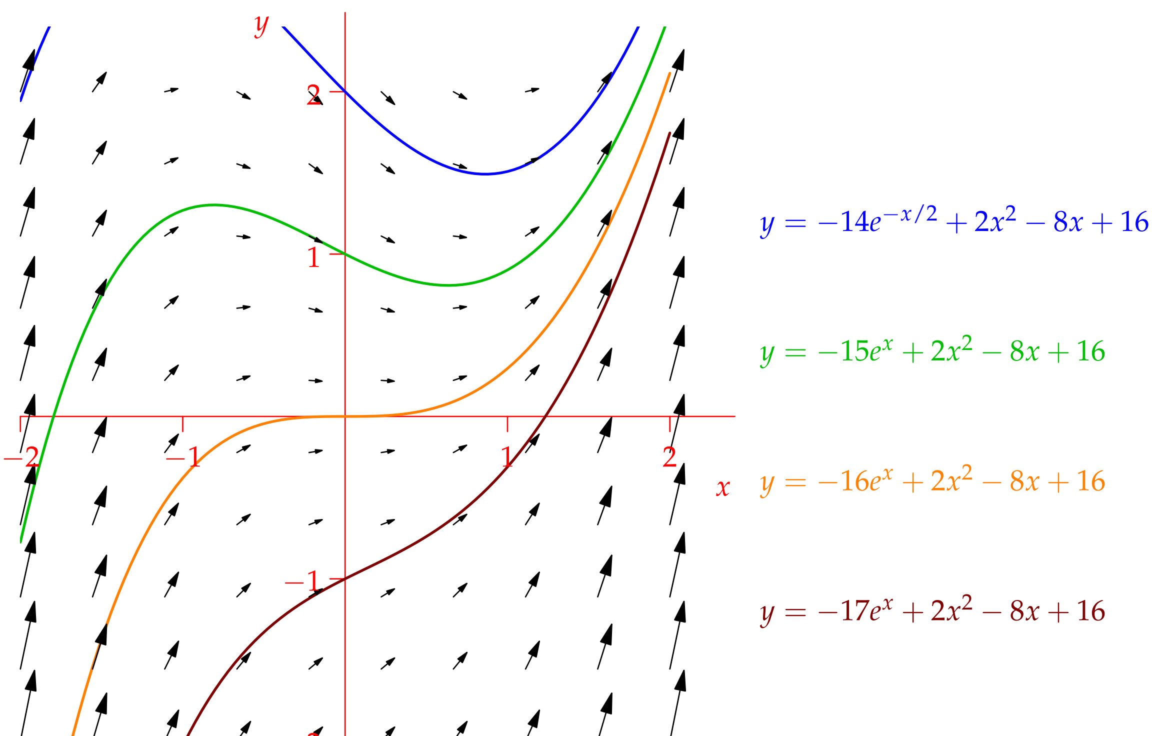
1. $\frac{dy}{dx} = y$ has solutions $y = ce^x$ where c is any constant



2. $\frac{dy}{dx} = y - x$ has solutions $y = ce^x + x + 1$



3. $\frac{dy}{dx} = x^2 - \frac{1}{2}y$ has solutions $y = ce^{-x/2} + 2x^2 - 8x + 16$



Euler's Method (non-examinable)

The slope field method is essentially how computers "solve" differential equations. Here is a very basic approach to finding an approximate solution to a 1^{st}-order initial value problem

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(a) = b \end{cases}$$

1. Apply Picard's Theorem to argue that the IVP has a unique solution.
2. Choose a *step size* h: a small positive number.
3. Place a dot at the point $P_0 := (a,b)$, the initial condition.
4. Move along the vector $h\begin{pmatrix} 1 \\ f(P_0) \end{pmatrix}$ and place a new point $P_1 := P_0 + h\big(1, f(P_0)\big)$.
5. Repeat for as many steps at you like.
6. Join the dots!

At each point P_j, the line segment $\overline{P_jP_{j+1}}$ is tangent to the slope field. The resulting piecewise curve *approximates* the solution to the original problem. For a more accurate approximation, increase the step size (and computing time).

The difficulties and challenges of Euler's method (and its commonly used "Runge–Kutta" extensions) are a matter for a more advanced course.

The pictures below show the method applied to the ODE $\frac{dy}{dx} = 2\sin(xy)$ with initial condition $y(-1) = 1$ and decreasing step sizes: $h_1 = 0.3$, $h_2 = 0.1$, $h_3 = 0.05$ and $h_4 = 0.025$. It certainly appears that as one reduces the step size, the approximate solution seems to be approaching something…