Forced Oscillations and Resonance (Section 2.6)

Consider the spring (oscillator) equation with external force $F(t)$. A basic case assumes F periodic,

$$mx'' + cx' + kx = F_0 \cos \omega t$$

Many real-life situations can be modeled this way, for example buildings in an earthquake, where the ground shakes periodically. In the abstract, there are two generic cases.

Case 1: Undamped, Driven Motion

The equation is $mx'' + kx = F_0 \cos \omega t$ (no friction $c = 0$). In the absence of an external force, the spring oscillates with circular frequency $\omega_0 = \sqrt{\frac{k}{m}}$: this provides the *complementary function*

$$x_C(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

It remains to find a suitable particular integral $x_P(t)$. For this there are two sub-cases.

Case 1(a): Beating ($\omega \neq \omega_0$) The most likely possibility is that the frequency of the external force $F_0 \cos \omega t$ differs from the natural frequency of the spring. We try a particular integral of the form

$$x_P(t) = a \cos \omega t + b \sin \omega t$$

Substituting into the differential equation, we see that

$$mx_P'' + kx_P = (-ma\omega^2 + ka) \cos \omega t + (-mb\omega^2 + kb) \sin \omega t = F_0 \cos \omega t$$

$$\implies a = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad b = 0$$

from which the general solution is

$$x(t) = x_P(t) + x_C(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

The result is a combination of two periodic motions with *different frequencies*. Larger F_0 and having ω close to ω_0 produce more motion. For simplicity, suppose our initial conditions are $x(0) = 0 = x'(0)$ (spring starts at rest). A little algebra and a trigonometric identity yield

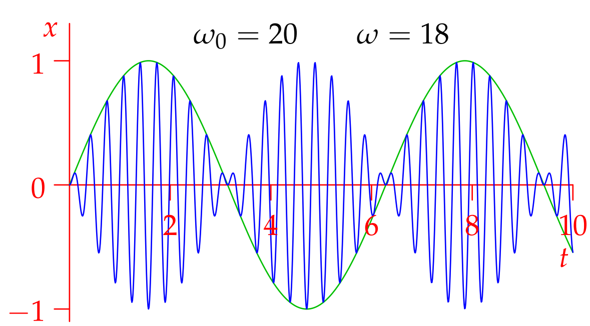
$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos \omega t - \cos \omega_0 t \right) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{\omega_0 - \omega}{2} t \sin \frac{\omega_0 + \omega}{2} t$$

If ω_0, ω are close in value, then $\omega_0 - \omega \ll \omega_0 + \omega$, and we have a high-frequency ($\frac{\omega_0 + \omega}{4\pi}$ Hz) oscillation whose amplitude varies at a lower frequency ($\frac{|\omega_0 - \omega|}{4\pi}$ Hz). This phenomenon is known as *beating*: a sound wave like this changes in volume.

Example A spring with $x'' + 400x = 38 \cos 18t$ has natural frequency $\omega_0 = \sqrt{400} = 20$ rad/s and driving frequency $\omega = 18$ rad/s.

The solution with initial conditions $x(0) = 0 = x'(0)$ is $x(t) = \sin t \sin 19t = A(t) \sin 19t$: a high frequency vibration $\sin 19t$ with periodic amplitude $A(t) = \sin t$.

Remember that this is a graph of the *extension* $x(t)$ of the spring, not a picture of the spring itself! To see the latter, see the animations here.



Case 1(b): Resonance ($\omega = \omega_0$) In this special subcase the driving frequency ω equals the natural frequency ω_0. The obvious guess from the previous subcase already solves the homogeneous ODE, so we try multiplying by t:

$$x_P(t) = at\cos\omega_0 t + bt\sin\omega_0 t$$

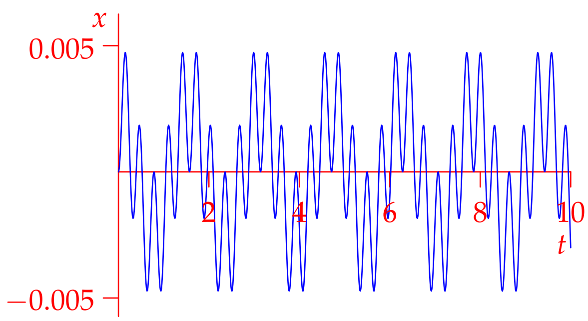
Substituting into the ODE and solving for a, b yields the general solution

$$x(t) = x_P(t) + x_C(t) = \frac{F_0}{2m\omega_0}t\sin\omega_0 t + c_1\cos\omega t + c_2\sin\omega_0 t$$

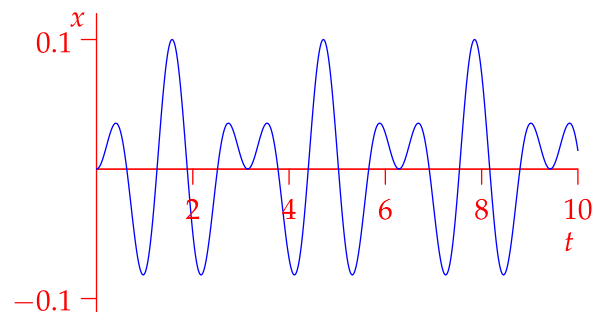
$x_P(t)$ is also the particular solution satisfying the initial conditions $x(0) = 0 = x'(0)$.

- $x_P(t)$ is a sine wave with *ever-increasing amplitude* $\frac{F_0}{2m\omega_0}t$: eventually the spring tears itself apart!
- The resonance solution is the limit $\lim\limits_{\omega\to\omega_0} x_P(t)$ of the beating solutions from subcase 1(a).

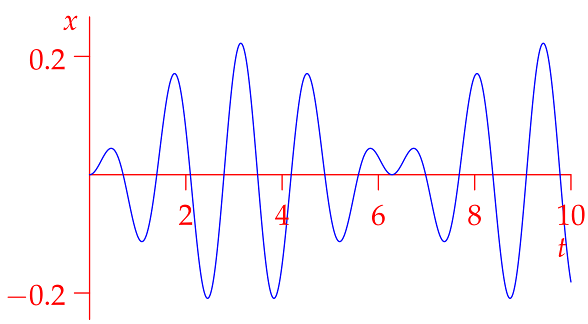
Examples Consider the initial value problem $x'' + 16x = \cos\omega t$, $x(0) = x'(0) = 0$. The natural frequency is $\omega_0 = 4$ rad/s. The graphs depict the solutions for four different driving frequencies ω: the final case is *resonance*. Animations showing how these examples change with ω are here.



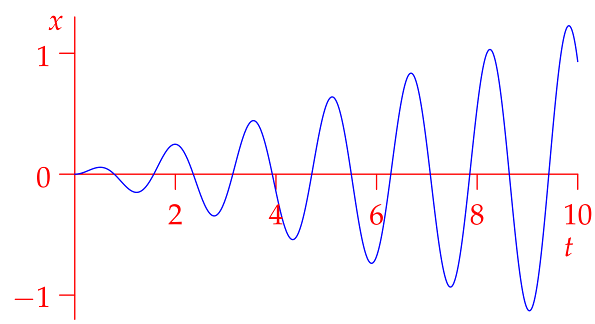
$\omega = 20 \quad x(t) = \frac{1}{192}\sin 8t\sin 12t$



$\omega = 6 \quad x(t) = \frac{1}{10}\sin t\sin 5t$



$\omega = 5 \quad x(t) = \frac{2}{9}\sin\frac{t}{2}\sin\frac{9}{2}t$



$\omega = 4 \quad x(t) = \frac{1}{8}t\sin 4t$

Summary of undamped, driven motion The initial value problem $mx'' + kx = F_0\cos\omega t$ with initial conditions $x(0) = 0 = x'(0)$ has solution

$$x(t) = \begin{cases} \frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{\omega_0-\omega}{2}t\sin\frac{\omega_0+\omega}{2}t & \text{if } \omega \neq \omega_0 \\ \frac{F_0}{2m\omega_0}t\sin\omega_0 t & \text{if } \omega = \omega_0 \end{cases}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. As $\omega \to \omega_0$, low frequency beats of increasing amplitude occur.

Case 2: Damped-driven motion & practical resonance

In the presence of damping $c > 0$, the complementary function is the solution to the free oscillator

$$mx'' + cx' + kx = F_0 \cos \omega t \implies x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

Damping type	Condition	Complementary Function $x_C(t)$
Over-damping	$c^2 > 4km$	$e^{-pt}\left(c_1 e^{-\sqrt{p^2-\omega_0^2}\,t} + c_2 e^{\sqrt{p^2-\omega_0^2}\,t}\right)$
Critical damping	$c^2 = 4km$	$(c_1 + c_2 t)e^{-pt}$
Under-damping	$c^2 < 4km$	$e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t)$ where $\omega_1 = \sqrt{\omega_0^2 - p^2}$

Regardless of initial conditions:

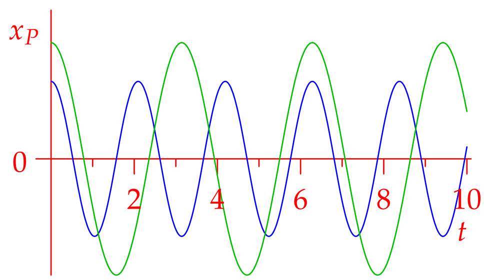
- $p = \frac{c}{2m} > 0 \implies x_C$ is *transient*: $\lim\limits_{t\to\infty} x_C(t) = 0$.
- For large t, we have $x(t) \approx x_P(t) = a \cos \omega t + b \sin \omega t$ is a *steady-periodic solution.*

Example Find the steady-periodic solution for the damped-driven oscillator $x'' + 2x' + 6x = \cos \omega t$. Substitute $x_P(t) = a \cos \omega t + b \sin \omega t$ into the ODE:

$$(-a\omega^2 + 2b\omega + 6a) \cos \omega t + (-b\omega^2 - 2a\omega + 6b) \sin \omega t = \cos \omega t$$

Solving for a and b produces the steady-periodic solution

$$\begin{aligned} x_P(t) &= \frac{1}{(\omega^2-6)^2 + 4\omega^2}\left((\omega^2 - 6)\cos \omega t + 4\omega \sin \omega t\right) \\ &= \frac{1}{\sqrt{(\omega^2-6)^2 + 4\omega^2}} \cos(\omega t - \gamma) \\ &= \frac{1}{\sqrt{(\omega^2-4)^2 + 20}} \cos(\omega t - \gamma) \end{aligned}$$



$x_P(t) = \frac{1}{3\sqrt{5}} \cos 3t$ and $\frac{1}{2\sqrt{5}} \cos 2t$

where γ is the phase angle. Note how the amplitude depends on the driving frequency ω: solutions are drawn with $\omega = 3$ and $\omega = 2$ rad/s. This last, the *maximum amplitude* solution, is an example of a general phenomenon known as *practical resonance.*

General situation The long-term, steady-periodic solution is

$$x_P(t) = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + 4p^2\omega^2}} \cos(\omega t - \gamma)$$

The amplitude is a function of the driving frequency ω. For fixed F_0, maximum amplitude occurs when the denominator is minimal: a little algebra shows that this is when

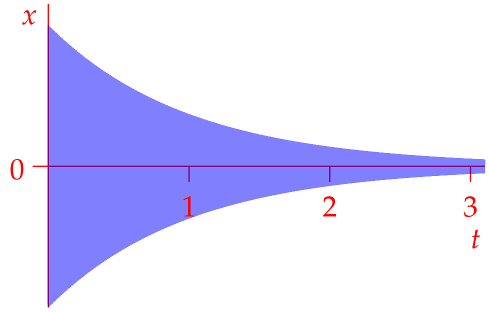
$$\omega = \begin{cases} \sqrt{\omega_0^2 - 2p^2} & \text{if } 2p^2 < \omega_0^2 \quad (\text{equivalently } c^2 < 2km) \\ 0 & \text{if } 2p^2 \ge \omega_0^2 \quad (c^2 \ge 2km) \end{cases}$$

Setting $\omega = \sqrt{\omega_0^2 - 2p^2}$ to maximize amplitude is *practical resonance.*

Example: Glass smashing! Model the transverse motion of the lip of a wine glass by the equation[1]

$$\tfrac{1}{10}x'' + \tfrac{1}{5}x' + 1{,}000{,}000x = F_0\cos\omega t$$

$F_0\cos\omega t$ models vibration of the air due to ambient sound. If one taps the glass, causing it to ring, the resulting motion is the complementary function



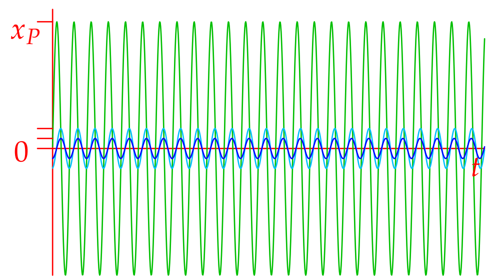
$$x_C(t) = e^{-t}(c_1\cos\omega_1 t + c_2\sin\omega_1 t)$$

with frequency

$$\frac{\omega_1}{2\pi} = \frac{\omega_1}{2\pi}\sqrt{\omega_0^2 - p^2} = \frac{\omega_1}{2\pi}\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = 503.2920959\,\text{Hz}$$

$\approx$ 1 octave above middle C. The plot of this motion is almost impossible to visualize due to its high frequency. Driving the oscillator models subjecting the glass to a loud tone of frequency $f = \frac{\omega}{2\pi}$: practical resonance occurs when

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\omega_0^2 - 2p^2} = 503.2920707\,\text{Hz}$$



Note how close this is to the natural frequency of the glass! The response of the glass is plotted if one supplies a tone at the resonant frequency, and both 1 and 2 Hz either side. You have to have a very good musical ear to notice a difference of 1 Hz at pitches like this: a singer must be both very loud and very accurate if they want to crack the glass…

Application: RLC circuits and Electrical Resonance

An RLC circuit has a source voltage[2] $V(t) = V_0\sin\omega t$, a resistor R Ohms, a capacitor C Farads, and an inductor L Henries connected in series.

The current flow in the circuit is $I = \frac{dQ}{dt}$ amperes, where Q (coulombs) is the charge stored in the capacitor at time t. By summing the voltage drop across each component, we obtain an ODE:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = V(t) \implies L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V'(t)$$

This is the *damped-driven* spring equation in disguise: $(m, c, k, F_0) \longleftrightarrow (L, R, C^{-1}, \omega V_0)$. The amplitude of the steady-periodic current

$$I = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

is plainly maximized when $\omega^2 LC = 1$: this is known as *electrical resonance*, a concept that has many applications in electronics and electrical engineering.

[1]Small mass, small damping, high spring constant

[2]For instance, US mains electricity is approximately $V(t) = 110\sqrt{2}\sin(120\pi t)$

Audio Hum Background noise is often related to an electrically resonant current. Hum can be reduced by adjusting the capacitance C to be very different to $\omega^{-2}L^{-1}$.

Reducing power loss If $C = 0$, the current flow is $I(t) = \frac{V_0}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t - \gamma)$. An inductor reduces (alternating) current flow, acting like extra resistance. By inserting a capacitor $C = \omega^{-2}L^{-1}$ into the circuit, the current flow increases to what one would expect ($I = V_0/R$) thus reducing power loss from natural inductance.[3]

Tuning an (AM) radio A radio station broadcasting at frequency ω induces a voltage $V(t) = V_0 \sin \omega t$ in a radio antenna, resulting in a current flow which (after amplification) powers a loudspeaker.

An old AM radio dial is a variable capacitor. By tuning it to $C = \omega^{-2}L^{-1}$, the signal with frequency ω is amplified, and current flows induced by other radio frequencies are diminished.

[3]As an electrical component, an inductor is often just a tight coil of wire: with lots of wiring, large electrical systems can easily form natural inductors!