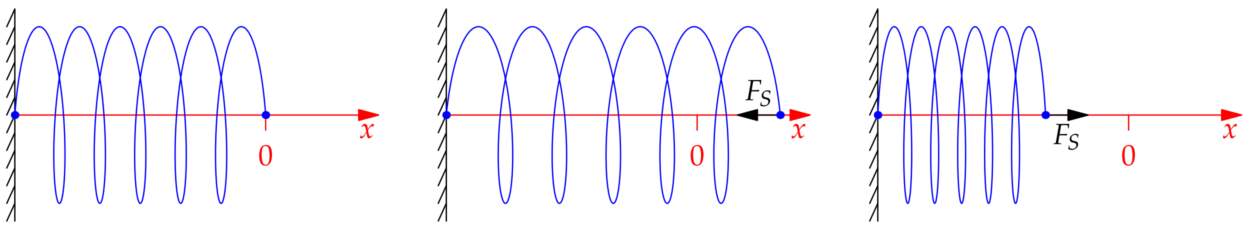
Mechanical Vibrations (Section 2.4)

This section contains lots of Physics language and notation. Don't try to memorize everything. Beyond basic modeling, treat questions on this material as open-book.

A mass m (kg) is attached to a spring. $x(t)$ measures distance (m) to the right of the equilibrium point at time t (s). A *spring force* F_S acts on the mass: provided the extension of the spring is small relative the size of the spring, experiments suggest that $F_S = -kx$ is a constant multiple of the extension: $k > 0$ is the *spring constant* (N/m).



Equilibrium: $x = 0, F_S = 0$ Extended: $x > 0, F_S < 0$ Compressed: $x < 0, F_S > 0$

Two other forces might act on the mass.

1. F_R = resistive force. For simplicity, this is often modeled as $F_R = -cx'$ where $c > 0$ is constant (Ns/m=kg/s). In reality, this isn't a very accurate model for friction/resistance, particularly when the mass is moving quickly.
2. $F_E = F(t)$, some time-dependent external force.

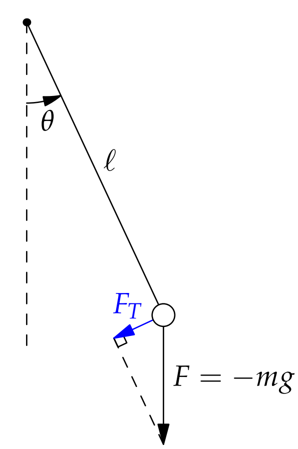
Newton's second law provides a a constant coefficient second-order linear ODE: $F = mx'' = F_S + F_R + F_E$ simplifies to the *spring equation*

$$mx'' + cx' + kx = F(t)$$

The motion of the mass/spring system is described as:

- **Forced** if $F(t) \not\equiv 0$, and **unforced** or **free** if $F(t) \equiv 0$. In this section, all motion will be unforced.
- **Damped** if $c > 0$, and **undamped** if $c = 0$ ($c < 0$ is unphysical).

This equation has other physical applications. For instance:



1. Applying Newton's second law to the angular component of gravity, we see that the motion of a pendulum satisfies the non-linear ODE

   $$m\ell\theta'' = F_T = -mg\sin\theta$$

   where g is the gravitational constant, and l is the length of the pendulum. If θ is small, then $\sin\theta \approx \theta$ and we obtain $\theta'' + \frac{g}{l}\theta = 0$.
2. The current $I(t)$ flowing in an RLC circuit (resistor, inductor, capacitor) may be modeled by the equation $LI' + RI + \frac{1}{C}\int I\,dt = V(t)$, where $V(t)$ is the applied voltage. Differentiate this to obtain $LI'' + RI' + \frac{1}{C}I = V'(t)$.

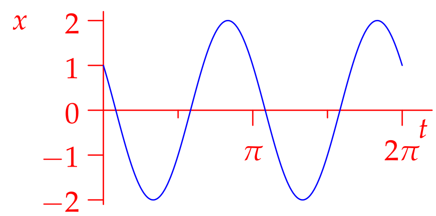
Free undamped motion ($F(t) = 0$, $c = 0$, $mx'' + kx = 0$)

The characteristic equation $m\lambda^2 + k = 0$ has complex roots $\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$ where $\omega_0 = \sqrt{\frac{k}{m}}$ (rad/s) is the *circular frequency*. The general solution

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \gamma)$$

is *simple harmonic motion* with *amplitude* $C = \sqrt{A^2 + B^2}$ (m) and *phase angle* $\gamma = \tan^{-1}\frac{B}{A}(+\pi)$ (rad). The motion is periodic, with *period* $T = \frac{2\pi}{\omega_0}$ (s) and *frequency* $f = \frac{\omega_0}{2\pi}$ (Hz $= \text{s}^{-1}$)

Examples 1. Suppose $k = 8$ N/m and $m = 2$ kg, and that the spring is set in motion with an extension of $x(0) = 1$ m and initial speed $x'(0) = -2\sqrt{3}$ m/s.



The general solution is $x(t) = A\cos 2t + B\sin 2t$. Apply the initial conditions to obtain

$$x(t) = \cos 2t - \sqrt{3}\sin 2t = 2\cos(2t + \tfrac{\pi}{3})$$

2. (Ex 2.4.4 from book.) A spring has $k = 4$ N/m. Suppose a mass is attached the the spring and set in motion. If the observed frequency is 0.8 Hz, find the mass.

   Since $f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, we see that

$$m = \frac{k}{4\pi^2 f^2} = \left(\frac{5}{4\pi}\right)^2 \approx 0.158 \text{ kg}$$

Free damped motion ($F(t) = 0$, $c > 0$, $mx'' + cx' + kz = 0$)

The equation may be re-written

$$x'' + 2px' + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped circular frequency and $p = \frac{c}{2m} > 0$. The characteristic equation is easily solved:

$$\lambda^2 + 2p\lambda + \omega_0^2 = 0 \Longrightarrow r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$$

There are three cases, dependent on the sign of the expression $p^2 - \omega_0^2$ in the square-root.

1. **Over-damping**: $p^2 - \omega_0^2 > 0$. The damping force $F_R = -cx'$ is large compared to the spring stiffness/mass ($c^2 > 4km$).

   Since $p > 0$, both roots r_1, r_2 are *real* and *negative*. The general solution is

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

   Regardless of the initial conditions, the motion of the spring dies away with time: $\lim_{t\to\infty} x(t) = 0$.

2. **Critical damping**: $p^2 - \omega_0^2 = 0$. Damping exactly balances the spring stiffness/mass ($c^2 = 4km$). The repeated root $r_1 = r_2 = -p$ is *real* and *negative*. The general solution is

$$x(t) = (c_1 + c_2 t)e^{-pt}$$

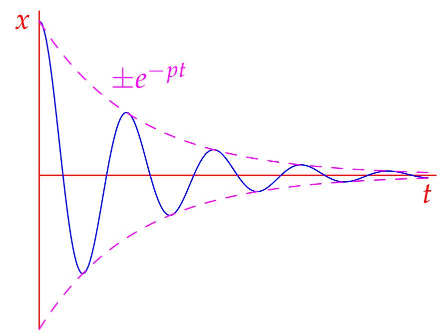
   Again, the motion of the spring dies away with time: $\lim_{t\to\infty} x(t) = 0$.

3. **Under-damping**: $p^2 - \omega_0^2 < 0$. Damping is small compared to the stiffness/mass ($c^2 < 4km$).

   The complex roots $r_1, r_2 = -p \pm i\sqrt{\omega_0^2 - p^2} = -p \pm i\omega_1$ have *negative real part*. The general solution is

$$\begin{aligned} x(t) &= e^{-pt}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) \\ &= Ce^{-pt}\cos(\omega_1 t - \gamma) \end{aligned}$$

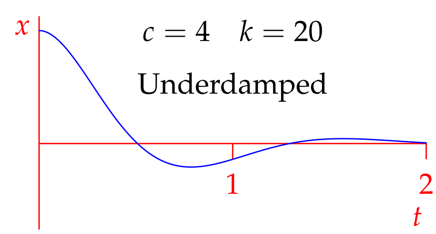
   where $C = \sqrt{A^2 + B^2}$ and γ is the phase angle. Solutions oscillate as they diminish to zero, but the modified frequency ω_1 is *smaller* than the natural frequency ω_0 of the equivalent undamped spring.



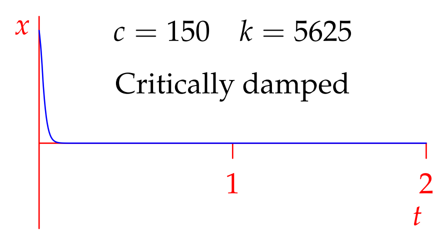
Suspension Examples

Vehicle suspensions may be modeled by these equations: in each example below, $m = 1$ models the vehicle's mass, k is supplied by the spring and c comes from the hydraulic dampers. Tuning the suspension of a vehicle means altering k and c.

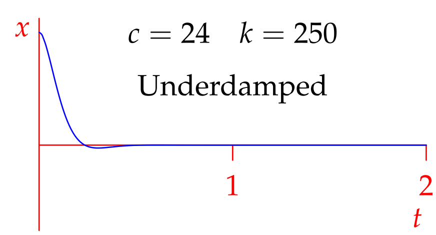
Tractors and Semi-trucks are usually *under-damped*: c and k are small, and $c^2 < 4km$ for a slow, relaxed response. This is ideal for traveling over rough ground or to not risk damaging cargo.



Sports car suspensions are typically closer to *critical damping*: c and k are very large, and $c^2 \approx 4km$ for a fast, stiff ride. Sports cars ride low to the ground for aerodynamics and so cannot bounce around. Tires also need to quickly be forced back to the road after going over a bump lest the vehicle lose grip and crash.



Family sedans are often *slightly under-damped*: c and k are moderate, with $c^2 < 4km$ for smooth but not bouncy response. Preferences have changed over time: look at a 1960s movie car chase to see how much bouncier normal cars were in the past!



For a given vehicle, increasing k results in a faster response, but the ride becomes more bouncy and shaky. Increasing c produces a slower response, and a softer, smoother ride.