# Math 4 Homework 2 Answers



§8.1

5/ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & y & z \\ 0 & y & z \\ y & 1 & z \end{bmatrix}$ for no values of $y, z$. It's impossible!

§8.2 2/a/ Cannot + or − as matrices different sizes

b/ $A+B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A-B = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

3/ $\underline{a}^T\underline{b} = (1\,2\,0)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1$, $\underline{b}\,\underline{a}^T = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}(1\,2\,0) = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

4/a/ $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}$

b/ $AB = \begin{pmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

5/ Profit = Income − Expenses

$= \underline{p}^T\underline{q} - \underline{w}^T\underline{z} = (10\;\,12)\begin{pmatrix} 15 \\ 27 \end{pmatrix} - (10\;\,10\;\,8)\begin{pmatrix} 11 \\ 15 \\ 15 \end{pmatrix}$

$= 150 + 324 - 110 - 150 - 120 = 94{,}000$

§8.3/ 2a/ Yes

b/ No − incompatible sizes of matrix

c/ No − " " " "

d/ No − " " " "

e/ No − cannot multiply $(2\times3)\times(2\times2)$.

5/ $\underset{(3\times5)\cdot(?)}{A \cdot B} = \underset{(3\times7)}{AB} \Rightarrow B$ is $5\times7$.

6/ $AB = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 21 & -6 \end{pmatrix}$

$AC = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 21 & -6 \end{pmatrix}$

7/ B has 2 rows.

§8.4

3/ $A^2 = \begin{pmatrix} x & -x \\ x-1 & 1-x \end{pmatrix}\begin{pmatrix} x & -x \\ x-1 & 1-x \end{pmatrix} = \begin{pmatrix} x^2 - x(x-1) & -x^2 - x(1-x) \\ x(x-1) + (1-x)(x-1) & -x(x-1) + (1-x)^2 \end{pmatrix}$

$= \begin{pmatrix} x^2 - x^2 + x & -x^2 - x + x^2 \\ x^2 - x - 1 + 2x - x^2 & -x^2 + x + 1 - 2x + x^2 \end{pmatrix} = \begin{pmatrix} x & -x \\ x-1 & 1-x \end{pmatrix} = A$

$\therefore$ A is idempotent.

5/ a/ Check : $A^2 = \begin{pmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{pmatrix}^2 = \frac{1}{36}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}^2 = \frac{1}{36}\begin{pmatrix} 6 & -12 & 6 \\ -12 & 24 & -12 \\ 6 & -12 & 6 \end{pmatrix} = A$

$\therefore$ A is idempotent and $AA = A^2 = A = AAA$.

$\therefore \operatorname{tr} A = \operatorname{tr}(AA) = \operatorname{tr}(AAA) = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1.$

b/ By 2/, A is idempotent $\therefore \operatorname{tr} A = \operatorname{tr}(AA) = \operatorname{tr}(AAA) = \frac{1}{11}(6+8+6+2) = 2.$

Ch Review

1/ a/ $AB$ is $2 \times 2$, b/ $A^T B$ is a scalar

c/ $A^T B A$ is a scalar, d/ $A A^T B$ is $5 \times 5$.

5/ $AB = \begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 7 & 4 \\ 5 & k \end{pmatrix} = \begin{pmatrix} 1 & 12-4k \\ -30 & k-20 \end{pmatrix}$

$BA = \begin{pmatrix} 7 & 4 \\ 5 & k \end{pmatrix}\begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -24 \\ 15-5k & k-20 \end{pmatrix}$

$\therefore AB = BA \iff \begin{cases} 15 - 5k = -30 \\ \& \\ 12 - 4k = -24 \end{cases} \iff \begin{cases} 5k = 15 \\ k \\ 4k = 12 \end{cases} \iff k = 3.$

§4.1

1/ a/ $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}$, b/ $\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$, c/ $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^{-1}$ does not exist.

~~3/ a/~~ $\cancel{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

$\cancel{\therefore z_1 = 2,\ z_2 = 1}$

~~b/~~ $\cancel{w^T A y = (5\ 10)\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}(}$

3a/ $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 20 \\ 55 \end{pmatrix} \Rightarrow z_1 = 20,\ z_2 = 55.$

b/ $\underline{w}^T A y = \underline{w}^T \underline{z}$ = total cost of inputs (scalar).

4/ a/ $\begin{pmatrix} D_c \\ D_s \end{pmatrix} = \begin{pmatrix} 100 \\ 80 \end{pmatrix} - \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} p_c \\ p_s \end{pmatrix}$, $\begin{pmatrix} S_c \\ S_s \end{pmatrix} = \begin{pmatrix} -20 \\ -10 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_c \\ p_s \end{pmatrix}$

Equilibrium: $\begin{pmatrix} D_c \\ D_s \end{pmatrix} = \begin{pmatrix} S_c \\ S_s \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -20 \\ -10 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_c \\ p_s \end{pmatrix} = \begin{pmatrix} 100 \\ 80 \end{pmatrix} - \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} p_c \\ p_s \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 7 & 1 \\ 2 & 5 \end{pmatrix} \underline{p} = \begin{pmatrix} 120 \\ 40 \end{pmatrix} = \underline{b} \qquad \left(A = \begin{pmatrix} 7 & 1 \\ 2 & 5 \end{pmatrix}\right)$$

b/ $$\underline{p} = \begin{pmatrix} 7 & 1 \\ 2 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 120 \\ 40 \end{pmatrix} = \frac{1}{33}\begin{pmatrix} 5 & -1 \\ -2 & 7 \end{pmatrix}\begin{pmatrix} 120 \\ 40 \end{pmatrix} = \frac{1}{33}\begin{pmatrix} 510 \\ 390 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 170 \\ 130 \end{pmatrix}$$

$\therefore$ Price coffee $= \frac{170}{11}$, Sugar $= \frac{130}{11}$.

§A.2/ 1/ a/ $\det A = -4 + 0 + 40 - 30 - 0 - 0 = 14$

b/ $\det A = 2\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 4\begin{vmatrix} 4 & 0 \\ 5 & 1 \end{vmatrix} = -10 + 3 - 16 = -23$

2/ a/ $M_{11} = \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} = -13$, $M_{12} = \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} = -2$, $M_{13} = \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 10$

$M_{21} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = -20$, $M_{22} = \begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} = -3$, $M_{23} = \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} = 15$

$M_{31} = \begin{vmatrix} 0 & 4 \\ 3 & 2 \end{vmatrix} = -12$, $M_{32} = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = -2$, $M_{33} = \begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = 9$

$$\therefore A^{-1} = \frac{1}{|A|}\operatorname{adj}A = \frac{1}{1}\begin{pmatrix} -13 & 20 & -12 \\ 2 & -3 & 2 \\ 10 & -15 & 9 \end{pmatrix}$$

b/ $$(C|I) = \left(\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 4 & 0 & 5 & 0 & 1 & 0 \\ 5 & 1 & 6 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ \rightsquigarrow -2R_1 \\ -2R_1 \end{array} \left(\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 0 & -6 & 13 & -2 & 1 & 0 \\ 1 & -5 & 14 & -2 & 0 & 1 \end{array}\right) \begin{array}{c} \rightsquigarrow -2R_3 \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 0 & 13 & -32 & 5 & 0 & -2 \\ 0 & -6 & 13 & -2 & 1 & 0 \\ 1 & -5 & 14 & -2 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \begin{array}{c} +2R_2 \\ \\ -R_2 \end{array} \left(\begin{array}{ccc|ccc} 0 & 1 & -6 & 1 & 2 & -2 \\ 0 & -6 & 13 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{c} \\ \rightsquigarrow +6R_1 \\ -R_1 \end{array} \left(\begin{array}{ccc|ccc} 0 & 1 & -6 & 1 & 2 & -2 \\ 0 & 0 & -23 & 4 & 13 & -12 \\ 1 & 0 & 7 & -1 & -3 & 3 \end{array}\right)$$

$$\rightsquigarrow \div -23 \left(\begin{array}{ccc|ccc} 0 & 1 & -6 & 1 & 2 & -2 \\ 0 & 0 & 1 & -4/23 & -13/23 & 12/23 \\ 1 & 0 & 7 & -1 & -3 & 3 \end{array}\right) \begin{array}{c} +6R_2 \\ \rightsquigarrow \\ -7R_2 \end{array} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -1/23 & -32/23 & 26/23 \\ 0 & 0 & 1 & -4/23 & -13/23 & 12/23 \\ 1 & 0 & 0 & 5/23 & 22/23 & -15/23 \end{array}\right)$$

$$\therefore C^{-1} = \frac{1}{23}\begin{pmatrix} 5 & 22 & -15 \\ -1 & -32 & 26 \\ -4 & -13 & 12 \end{pmatrix}$$

[ Use whichever method works for you ]

4/ $\det C = -\det A = -3$ since swapped $2^{nd}$ & $3^{rd}$ rows of A.

5/ $\det D = 3\det A = 9$. Adding a, b, c to $2^{nd}$ row is irrelevant.
Mult $2^{nd}$ row by $3 \to$ mult det by 3.

6/ $\det\begin{bmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{bmatrix} = -175 + 0 - 196 - 0 + 160 + 210 = -1$

$\neq 0 \Rightarrow$ vectors linearly independent.

7/ $\det A = 1 + 0 + 2 - 4 - 3 - 0 = -4$

$\therefore \det A^3 = (\det A)^3 = -64.$