Math 4 Homework 3

§9.4

1/b/ $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -5 & 2 & 3 \end{bmatrix}$, $M_{11} = 3, M_{12} = 0, M_{13} = 5$
$M_{21} = 8, M_{22} = -2, M_{23} = 12$
$M_{31} = 1, M_{32} = 0, M_{33} = 1$

$|A| = 3 - 0 - 5 = -2$

$$\therefore A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 3 & -8 & 1 \\ 0 & -2 & 0 \\ 5 & -12 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 & 8 & -1 \\ 0 & 2 & 0 \\ -5 & 12 & -1 \end{pmatrix}$$

2/b/ $$(A|I) = \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-3R_3} \left(\begin{array}{ccc|ccc} 0 & -6 & 1 & 1 & 0 & -3 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \begin{array}{c} +3R_2 \\ \\ -R_2 \end{array}\left(\begin{array}{ccc|ccc} 0 & 0 & 4 & 1 & 3 & -3 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \rightarrow \div 4 \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1/4 & 3/4 & -3/4 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 1 \end{array}\right)$$

$$\rightarrow \begin{array}{c} \\ -R_1 \\ R_1 \end{array}\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1/4 & 3/4 & -3/4 \\ 0 & 2 & 0 & -1/4 & 1/4 & 3/4 \\ 1 & 0 & 0 & 1/4 & -1/4 & 1/4 \end{array}\right) \rightarrow \begin{array}{c} R_3 \\ \div 2 \\ R_1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/4 & 1/4 \\ 0 & 1 & 0 & -1/4 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1/4 & 3/4 & -3/4 \end{array}\right)$$

5/c/ $$|C| = \begin{vmatrix} 1 & 0 & 4 \\ 2 & 8 & 5 \\ -1 & -2 & 3 \end{vmatrix} - \begin{vmatrix} -1 & -3 & 0 \\ 1 & 0 & 4 \\ -1 & -2 & 3 \end{vmatrix} + \begin{vmatrix} -1 & -3 & 0 \\ 1 & 0 & 4 \\ 2 & 8 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 5 \\ -2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 8 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 4 \\ -2 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 4 \\ 8 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}.$$

$$= 34 + 4(4) + 8 - 3(7) + 32 + 3(-3)$$

$$= 60.$$

6/ ~~$(ABC)(C^{-1}B^{-1}A^{-1})$~~

$$(ABC)^{-1} = ((AB)C)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}.$$

§9.4

1. $A\underline{x} = \underline{b}$ where $A = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix}$

$|A| = 0+0+0-(-2-6) = 8$

$x_1 = \frac{1}{8} \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = \frac{1}{8}(0-3+0-(7-16+0)) = 3/4,$

$x_2 = \frac{1}{8} \begin{vmatrix} -2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = \frac{1}{8}(32+0+0-(6-42+0)) = 17/2,$

$x_3 = \frac{1}{8} \begin{vmatrix} -2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = \frac{1}{8}(0+0-21-(0+16+9)) = \frac{-23}{4}.$

4(b) $\underline{x} = (I-A)^{-1}\underline{d} = \begin{pmatrix} 0.8 & -0.2 & -0.3 \\ -0.3 & 0.5 & -0.3 \\ -0.4 & -0.2 & 0.7 \end{pmatrix}^{-1} \begin{pmatrix} 10\,000 \\ 20\,000 \\ 40\,000 \end{pmatrix}$

$= \frac{1}{88} \begin{pmatrix} 29 & 20 & 21 \\ 33 & 44 & 33 \\ 26 & 24 & 34 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \times 10^5$

$= \frac{10^5}{88} \begin{pmatrix} 153 \\ 253 \\ 210 \end{pmatrix} = \begin{pmatrix} 173863.6364 \\ 287500 \\ 238636.3636 \end{pmatrix}$

Ch 9 Review

1. No, $\det A = 0$ $\left(\text{or } \begin{pmatrix} 12 \\ 15 \\ 18 \end{pmatrix} + \begin{pmatrix} 14 \\ 17 \\ 20 \end{pmatrix} = 2\begin{pmatrix} 13 \\ 16 \\ 19 \end{pmatrix} \right)$

6, Yes: $|AB| = |A|\cdot|B| = |B|\cdot|A| = |BA|$.

7, $|A^T A| = |I| \Rightarrow |A^T|\cdot|A| = 1$
$\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1.$

8, $|BAB^{-1}| = |B||A||B^{-1}| = |B||A|\frac{1}{|B|} = |A|.$

§10.1

1. $|\underline{y}| = \sqrt{6}$, $|\underline{w}| = \sqrt{8}$, $|\underline{z}| = 1$, $|\underline{v}| = 1/\sqrt{3}$

2/c/ $\underline{y}\cdot\underline{w} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1.$

3/b/ $\det(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{vmatrix} = 8 \neq 0 \Rightarrow$ lin indep.

4/b/ $\underline{z} = -\underline{e}_1 + 2\underline{e}_2 + \underline{e}_4$

$\quad = -\underline{v}_1 + \underline{v}_2 + \frac{1}{4}\underline{v}_4.$

8/ max rank $= 6$.

9/ rank $A = 3$ $\quad(\det A = -1 \neq 0)$

rank $B = 3$ (contains A on left & max rank $= 3$).

§10.2

1/ a/ $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0 \Leftrightarrow 2-\lambda = \pm 1$

$\Leftrightarrow \lambda = 1, 3.$

b/ $\underline{\lambda_1 = 1}$ $\quad (A - \lambda_1 I)\underline{v}_1 = \underline{0} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\underline{v}_1 = 0 \Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{\lambda_2 = 3}$ $\quad (A - 3I)\underline{v}_2 = 0 \Leftrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\underline{v}_2 = 0 \Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c/ $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}.$

3/ a/ $P^2 = (X(X^TX)^{-1}X^T)(X(X^TX)^{-1}X^T)$

$\quad = X(X^TX)^{-1}(X^TX)(X^TX)^{-1}X^T = X(X^TX)^{-1}X^T = P$

$\therefore$ P is idempotent.

b/ $P^2 = P$. Let $P\underline{x} = \lambda\underline{x}$. Then $\lambda^2\underline{x} = \lambda\underline{x} \Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 0, 1$

5/ Let P, Q be orthogonal, then

$$(PQ)^T(PQ) = Q^T P^T P Q = Q^T Q = I$$

$\therefore PQ$ orthogonal.

§10.3

1/a/ leading principal minors:

$$|A_1| = |1| = 1, \quad |A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 4 - 1 = 3$$

$$|A_3| = |A| = 12 + 0 + 0 - (0 + 4 + 3) = 5$$

All $> 0 \Rightarrow$ A is positive definite.

c/ $|C_1| = 1$, $|C_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$$|C_3| = |C| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2 + 0 + 0 - (1 + 0 + 1) = 0$$

All ≥ 0 ~~$\Rightarrow$ positive semi-definite~~.

Remaining principal minors:

$|1| = 1$, $|2| = 2$

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

All $\geq 0 \Rightarrow$ positive semi-definite.

b/ ~~leading principal minors~~:

~~$|B_1| \geq 5$, $|B_2| \geq \begin{vmatrix} 5 & -6 \\ -4 & 2 \end{vmatrix}$~~

$$\left.\begin{array}{l} (1,0,0)B\begin{pmatrix}1\\0\\0\end{pmatrix} = 5 > 0 \\ (0,0,1)B\begin{pmatrix}0\\0\\1\end{pmatrix} = -4 < 0 \end{array}\right\} \Rightarrow \text{indefinite}.$$

3/ $g(\underline{x}) = \underline{x}^T \begin{pmatrix} 5 & -\frac{1}{2} & 4 \\ -\frac{1}{2} & 3 & 0 \\ 4 & 0 & 2 \end{pmatrix} \underline{x}$

5/ $g(\underline{x}) = \underline{x}^T \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix} \underline{x} = \underline{x}^T A \underline{x}$

leading principal minors: $|A_1| = 3$, $|A_2| = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$

$|A_3| = |A| = 6 + 0 + 0 - (12 + 4 + 0) = -10$

$\therefore$ g is indefinite.

Alternatively, observe $g\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 > 0$ and $g\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1 < 0$.

<u>Ch 10 Review</u>

1/c/ $\det(\underline{y}, \underline{w}, \underline{z}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 0 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -3 \neq 0$

$\Rightarrow$ linearly independent.

2/ 3.

3/a/ 7, b/ 7 Rank = max # lin indep rows or columns, which must be $\leq$ # rows and columns.

4/ rank $A = 2$ (notice $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$)

rank $B = 3$ ($\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ lin indep).

7/ a/ $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (\lambda-2)(\lambda+6) - 9 = \lambda^2 + 4\lambda - 21 = (\lambda+7)(\lambda-3)$

$\therefore$ e'values are 3 and -7.

b/ $|B - \lambda I| = \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 49 = 0 \Leftrightarrow \lambda = 2 \pm 7 = 9, -5$

$\therefore$ e'values are $9, -5$.

8/ e'values are 3, 1, 2 – upper triangular matrices have e'values down diagonal.